ALGEBRA FOR BEGINNERS.

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BY

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REVISED AND ADAPTED TO AMERICAN SCHOOLS

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PREFACE.

THE rearrangement of the *Elementary Algebra* of Messrs. Hall and Knight was undertaken in the hope of being able to give to our advanced secondary schools a work that would fully meet their requirements in this important study. Many changes were made and additional subject-matter introduced. The *Algebra for Beginners*, however, so fully meets the needs of the class of students for which it was written, that we have made only such changes as seemed to bring out more clearly important points, and better adapt it to American schools.

With reference to the arrangement of topics, we quote from Messrs. Hall and Knight's preface to a former edition:

"Our order has been determined mainly by two considerations: first, a desire to introduce as early as possible the practical side of the subject, and some of its most interesting applications, such as easy equations and problems; and secondly, the strong opinion that all reference to compound expressions and their resolution into factors should be postponed until the usual operations of Algebra have been exemplified in the case of simple expressions. By this course the beginner soon

becomes acquainted with the ordinary algebraical processes without encountering too many of their difficulties; and he is learning at the same time something of the more attractive parts of the subject.

"As regards the early introduction of simple equations and problems, the experience of teachers favors the opinion that it is not wise to take a young learner through all the somewhat mechanical rules of Factors, Highest Common Factor, Lowest Common Multiple, Involution, Evolution, and the various types of Fractions, before making some effort to arouse his interest and intelligence through the medium of easy equations and problems. Moreover, this view has been amply supported by all the best text-books on Elementary Algebra which have been recently published."

The work will be found to meet the wants of all who do not require a knowledge of Algebra beyond Quadratic Equations — that portion of the subject usually covered in the examination for admission to the classical course of American Colleges.

FRANK L. SEVENOAK.

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ALGEBRA.

CHAPTER I.

DEFINITIONS. SUBSTITUTIONS.

1. ALGEBRA treats of quantities as in Arithmetic, but with greater generality; for while the quantities used in arithmetical processes are denoted by *figures* which have one single definite value, algebraical quantities are denoted by *symbols* which may have any value we choose to assign to them.

The symbols employed are letters, usually those of our own alphabet; and, though there is no restriction as to the numerical values a symbol may represent, it is understood that in the same piece of work it keeps the same value throughout. Thus, when we say "let $a = 1$," we do not mean that a must have the value 1 always, but only in the particular example we are considering. Moreover, we may operate with symbols without assigning to them any particular numerical value at all; indeed it is with such operations that Algebra is chiefly concerned.

We begin with the definitions of Algebra, premising that the symbols $+$, $-$, $\times$, $\div$, will have the same meanings as in Arithmetic.

2. An **algebraical expression** is a collection of symbols; it may consist of one or more **terms**, which are separated from each other by the signs $+$ and $-$. Thus $7a+5b-3c-x+2y$ is an expression consisting of five terms.

Note. When no sign precedes a term the sign $+$ is understood.

3. **Expressions** are either **simple** or **compound.** A *simple expression* consists of *one* term, as $5a$. A *compound expression* consists of *two or more* terms. Compound expressions may be

further distinguished. Thus an expression of *two* terms, as $3a-2b$, is called a **binomial** expression; one of *three* terms, as $2a-3b+c$, a **trinomial**; one of *more than three* terms a **multinomial.**

4. When two or more quantities are multiplied together the result is called the **product.** One important difference between the notation of Arithmetic and Algebra should be here remarked. In Arithmetic the product of 2 and 3 is written 2×3, whereas in Algebra the product of a and b may be written in any of the forms $a\times b$, $a.b$, or ab. The form ab is the most usual. Thus, if $a=2$, $b=3$, the product $ab=a\times b=2\times 3=6$; but in Arithmetic 23 means "twenty-three," or $2\times 10+3$.

5. Each of the quantities multiplied together to form a product is called a **factor** of the product. Thus 5, a, b are the factors of the product $5ab$.

6. When one of the factors of an expression is a numerical quantity, it is called the **coefficient** of the remaining factors. Thus in the expression $5ab$, 5 is the coefficient. But the word coefficient is also used in a wider sense, and it is sometimes convenient to consider any factor, or factors, of a product as the coefficient of the remaining factors. Thus in the product $6abc$, $6a$ may be appropriately called the coefficient of bc. A coefficient which is not merely numerical is sometimes called a **literal coefficient.**

Note. When the coefficient is unity it is usually omitted. Thus we do not write $1a$, but simply a.

7. If a quantity be multiplied by itself any number of times, the product is called a **power** of that quantity, and is expressed by writing the number of factors to the right of the quantity and above it. Thus

$a\times a$ is called the *second power* of a, and is written a^2;

$a\times a\times a$..............*third power* of a, ... a^3;

and so on.

The number which expresses the power of any quantity is called its **index** or **exponent.** Thus 2, 5, 7 are respectively the indices of a^2, a^5, a^7.

Note. a^2 is usually read "a squared"; a^3 is read "a cubed"; a^4 is read "a to the fourth"; and so on.

When the index is unity it is omitted, and we do not write a^1, but simply a. Thus a, $1a$, a^1, $1a^1$ all have the same meaning.

8. The beginner must be careful to distinguish between *coefficient* and *index*.

Example 1. What is the difference in meaning between $3a$ and a^3?

By $3a$ we mean the product of the quantities 3 and a.

By a^3 we mean the third power of a; that is, the product of the quantities a, a, a.

Thus, if $a = 4$,

$$3a = 3 \times a = 3 \times 4 = 12;$$

$$a^3 = a \times a \times a = 4 \times 4 \times 4 = 64.$$

Example 2. If $b = 5$, distinguish between $4b^2$ and $2b^4$.

Here $4b^2 = 4 \times b \times b = 4 \times 5 \times 5 = 100$;

whereas $2b^4 = 2 \times b \times b \times b \times b = 2 \times 5 \times 5 \times 5 \times 5 = 1250.$

Example 3. If $x = 1$, find the value of $5x^4$.

Here $5x^4 = 5 \times x \times x \times x \times x = 5 \times 1 \times 1 \times 1 \times 1 = 5.$

Note. The beginner should observe that every power of 1 is 1.

9. In arithmetical multiplication the order in which the factors of a product are written is immaterial. For instance 3×4 means 4 sets of 3 units, and 4×3 means 3 sets of 4 units; in each case we have 12 units in all. Thus

$$3 \times 4 = 4 \times 3.$$

In a similar way,

$$3 \times 4 \times 5 = 4 \times 3 \times 5 = 4 \times 5 \times 3;$$

and it is easy to see that the same principle holds for the product of any number of arithmetical quantities.

In like manner in Algebra ab and ba each denote the product of the two quantities represented by the letters a and b, and have therefore the same value. Again, the expressions abc, acb, bac, bca, cab, cba have the same value, each denoting the product of the three quantities a, b, c. It is immaterial in what order the factors of a product are written; it is usual, however, to arrange them in alphabetical order.

Fractional coefficients which are greater than unity are usually kept in the form of improper fractions.

Example 4. If $a = 6$, $x = 7$, $z = 5$, find the value of $\frac{13}{10}axz$.

Here $$\frac{13}{10}axz = \frac{13}{10} \times 6 \times 7 \times 5 = 273.$$

EXAMPLES I. a.

If $a=5$, $b=4$, $c=1$, $x=3$, $y=12$, $z=2$, find the value of

1. $2a$.	2. a^2.	3. $3z$.	4. z^3.	5. c^4.
6. $4c$.	7. $4b^2$.	8. c^3.	9. x^3.	10. $3x$.
11. $7y^2$.	12. $8a^3$.	13. $6z^5$.	14. $5z^6$.	15. $7c^6$.

If $a=6$, $p=4$, $q=7$, $r=5$, $x=1$, find the value of

16. ap.	17. $3pq$.	18. $3qx$.	19. $5p^3$.	20. $8aqx$.
21. pqr.	22. $8aqr$.	23. $7qrx$.	24. $2apx$.	25. $7x^4$.
26. $3p^4$.	27. $8r^4$.	28. $9apqx$.	29. $6x^7$.	30. x^{10}.

If $h=5$, $k=3$, $x=4$, $y=1$, find the value of

31. $\frac{1}{9}k^3$.	32. $\frac{1}{6}kx$.	33. $\frac{1}{7}y^7$.	34. $\frac{1}{10}hkx$.	35. $\frac{1}{10}y^5$.
36. $\frac{1}{8}x^4$.	37. $\frac{1}{27}k^5$.	38. $\frac{1}{125}h^6$.	39. $\frac{1}{6}y^8$.	40. $\frac{1}{8}hkxy$.

10. When several different quantities are multiplied together a notation similar to that of Art. 7 is adopted. Thus $aabbbbcddd$ is written $a^2b^4cd^3$. And conversely $7a^3cd^2$ has the same meaning as $7\times a\times a\times a\times c\times d\times d$.

Example 1. If $c=3$, $d=5$, find the value of $16c^4d^3$.

Here $16c^4d^3=16\times 3^4\times 5^3=(16\times 5^3)\times 3^4=2000\times 81=162000$.

Note. The beginner should observe that by a suitable combination of the factors some labour has been avoided.

Example 2. If $p=4$, $q=9$, $r=6$, $s=5$, find the value of $\dfrac{32qr^3}{81p^5}$.

Here
$$\frac{32qr^3}{81p^5}=\frac{32\times 9\times 6^3}{81\times 4^5}=\frac{32\times 9\times 6\times 6\times 6}{81\times 4\times 4\times 4\times 4\times 4}=\frac{3}{4}.$$

11. If one factor of a product is equal to 0, the product must be equal to 0, *whatever values the other factors may have.* A factor 0 is usually called a *zero factor.*

For instance, if $x=0$ then ab^3xy^2 contains a zero factor. Therefore $ab^3xy^2=0$ when $x=0$, whatever be the values of a, b, y.

Again, if $c=0$, then $c^3=0$; therefore $ab^2c^3=0$, whatever values a and b may have.

Note. Every power of 0 is 0.

EXAMPLES I. b.

If $a = 3$, $b = 2$, $p = 10$, $q = 1$, $x = 0$, $z = 7$, find the value of

1. $3bp$.	2. $8ax$.	3. $5pqz$.	4. $6aqz$.	5. bpz.
6. $3b^2q$.	7. az^2.	8. q^2x^3.	9. qz^2.	10. $5b^3px^4$.
11. a^3p^4x.	12. $8p^3q^5$.	13. b^5a^3	14. px^4z^3.	15. $8a^4q^5$.

If $k = 1$, $l = 2$, $m = 0$, $p = 3$, $q = 4$, $r = 5$, find the value of

16. $\dfrac{3k^2}{p^2}$.	17. $\dfrac{5l^3}{qr}$.	18. $\dfrac{m}{3k^2}$.	19. $\dfrac{3m^2}{4l}$.	20. $\dfrac{16p^3}{9q^2}$.
21. $\dfrac{5m^3}{k^5}$.	22. $\dfrac{6l^4}{3q^3}$.	23. $\dfrac{8r^3}{25lq^2}$.	24. $\dfrac{9mq}{4p^2}$.	25. $\dfrac{81q^4}{400l}$.
26. $\dfrac{m^q}{l^p}$.	27. $\dfrac{q^k}{l^r}$.	28. $\dfrac{kr^k}{q^l}$.	29. $\dfrac{5m^r}{k^p}$.	30. $\dfrac{k^r}{r^k}$.

12. We now proceed to find the numerical value of expressions which contain more than one term. In these each term can be dealt with singly by the rules already given, and by combining the terms the numerical value of the whole expression is obtained.

13. We have already, in Art. 8, drawn attention to the importance of carefully distinguishing between *coefficient* and *index*; confusion between these is such a fruitful source of error with beginners that it may not be unnecessary once more to dwell on the distinction.

Example. When $c = 5$, find the value of $c^4 - 4c + 2c^3 - 3c^2$.

Here
$$c^4 = 5^4 = 5 \times 5 \times 5 \times 5 = 625;$$
$$4c = 4 \times 5 = 20;$$
$$2c^3 = 2 \times 5^3 = 2 \times 5 \times 5 \times 5 = 250;$$
$$3c^2 = 3 \times 5^2 = 3 \times 5 \times 5 = 75.$$

Hence the value of the expression
$$= 625 - 20 + 250 - 75 = 780.$$

14. The beginner must also note the distinction in meaning between the *sum* and the *product* of two or more algebraical quantities. For instance, ab is the *product* of the two quantities a and b, and its value is obtained by *multiplying* them together. But $a + b$ is the *sum* of the two quantities a and b, and its value is obtained by *adding* them together.

Thus if $a=11$, $b=12$.

the *sum* of a and b is $11+12$, that is, 23;

the *product* of a and b is 11×12, that is, 132.

15. By Art. 11 any term which contains a *zero factor* is itself zero, and may be called a *zero term.*

Example. If $a=2$, $b=0$, $x=5$, $y=3$, find the value of

$$5a^3 - ab^2 + 2x^2y + 3bxy.$$

$$\text{The expression} = (5 \times 2^3) - 0 + (2 \times 5^2 \times 3) + 0$$
$$= 40 + 150 = 190.$$

Note. The two zero terms do not affect the result.

16. In working examples the student should pay attention to the following hints.

1. Too much importance cannot be attached to neatness of style and arrangement. The beginner should remember that neatness is in itself conducive to accuracy.

2. The sign $=$ should never be used except to connect quantities which are equal. Beginners should be particularly careful not to employ the sign of equality in any vague and inexact sense.

3. Unless the expressions are very short the signs of equality in the steps of the work should be placed one under the other.

4. It should be clearly brought out how each step follows from the one before it; for this purpose it will sometimes be advisable to add short verbal explanations; the importance of this will be seen later.

EXAMPLES I. c.

If $a=4$, $b=1$, $c=3$, $f=5$, $g=7$, $h=0$, find the value of

1. $3f+5h-7b$.
2. $7c-9h+2a$.
3. $4g-5c-9b$.
4. $3g-4h+7c$.
5. $3f-2g-b$.
6. $9b-3c+4h$.
7. $3a-9b+c$.
8. $2f-3g+5a$.
9. $3c-4a+7b$.
10. $3f+5h-2c-4b+a$.
11. $6h-7b-5a-7f+9g$.
12. $7c+5b-4a+8h+3g$.
13. $9b+a-3g+4f+7h$.
14. $fg+gh-ab$.
15. $gb-3hc+fb$.
16. $fh+hb-3hc$.
17. $f^2-3a^2+2c^3$.
18. $b^3-2h^3+3a^2$.
19. $3b^2-2b^3+4h^2-2h^3$

CHAPTER II.

NEGATIVE QUANTITIES. ADDITION OF LIKE TERMS.

17. IN his arithmetical work the student has been accustomed to deal with numerical quantities connected by the signs + and − ; and in finding the value of an expression such as $1\frac{3}{4}+7\frac{2}{3}-3\frac{1}{8}+6-4\frac{1}{5}$ he understands that the quantities to which the sign + is prefixed are *additive*, and those to which the sign − is prefixed are *subtractive*, while the first quantity, $1\frac{3}{4}$, to which no sign is prefixed, is counted among the additive terms. The same notions prevail in Algebra ; thus in using the expression $7a+3b-4c-2d$ we understand the symbols $7a$ and $3b$ to represent additive quantities, while $4c$ and $2d$ are subtractive.

18. In Arithmetic the sum of the additive terms is always greater than the sum of the subtractive terms ; if the reverse were the case the result would have no arithmetical meaning. In Algebra, however, not only may the sum of the subtractive terms exceed that of the additive, but a subtractive term may stand alone, and yet have a meaning quite intelligible.

Hence all algebraical quantities may be divided into **positive quantities** and **negative quantities**, according as they are expressed with the sign + or the sign − ; and this is quite irrespective of any actual process of addition and subtraction.

This idea may be made clearer by one or two simple illustrations.

(i) Suppose a man were to gain \$100 and then lose \$70, his total *gain* would be \$30. But if he first gains \$70 and then loses \$100 the result of his trading is a *loss* of \$30.

The corresponding algebraical statements would be

$$\$100-\$70 = +\$30,$$

$$\$70-\$100=-\$30,$$

and the negative quantity in the second case is interpreted as a *debt*, that is, a sum of money opposite in character to the positive quantity, or *gain*, in the first case; in fact it may be said to possess a subtractive quality which would produce its effect on other transactions, or perhaps wholly counterbalance a sum gained.

(ii) Suppose a man starting from a given point were to walk along a straight road 100 yards forwards and then 70 yards backwards, his distance from the starting-point would be 30 yards. But if he first walks 70 yards forwards and then 100 yards backwards his distance from the starting-point would be 30 yards, but *on the opposite side of it*. As before we have

$$100 \text{ yards} - 70 \text{ yards} = +30 \text{ yards},$$
$$70 \text{ yards} - 100 \text{ yards} = -30 \text{ yards}.$$

In each of these cases the man's *absolute distance* from the starting point is the same; but by taking the positive and negative signs into account, we see that -30 is a distance from the starting point *equal in magnitude but opposite in direction* to the distance represented by $+30$. Thus the negative sign may here be taken as indicating *a reversal of direction*.

(iii) The freezing point of the Centigrade thermometer is marked zero, and a temperature of 15° C. means 15° above the freezing point, while a temperature 15° below the freezing point is indicated by $-15°$ C.

19. Many other illustrations might be chosen; but it will be sufficient here to remind the student that a subtractive quantity is always opposite in character to an additive quantity of equal *absolute value*. In other words *subtraction is the reverse of addition.*

20. Definition. When terms do not differ, or when they differ only in their numerical coefficients, they are called **like**, otherwise they are called **unlike**. Thus $3a$, $7a$; $5a^2b$, $2a^2b$; $3a^3b^2$, $-4a^3b^2$ are pairs of like terms; and $4a$, $3b$; $7a^2$, $9a^2b$ are pairs of unlike terms.

Addition of Like Terms.

Rule I. *The sum of a number of like terms is a like term.*

Rule II. *If all the terms are positive, add the coefficients.*

Example. Find the value of $8a+5a$.

Here we have to increase 8 like things by 5 like things of the same kind, and the aggregate is 13 of such things;

for instance, $\quad$ 8 lbs. + 5 lbs. = 13 lbs.

Hence also, $$8a+5a=13a.$$

Similarly, $$8a+5a+a+2a+6a=22a.$$

Rule III. *If all the terms are negative, add the coefficients numerically and prefix the minus sign to the sum.*

Example. To find the sum of $-3x$, $-5x$, $-7x$, $-x$.

Here the word *sum* indicates the aggregate of 4 subtractive quantities of like character. In other words, we have to *take away* successively 3, 5, 7, 1 like things, and the result is the same as taking away $3+5+7+1$ such things in the aggregate.

Thus the sum of $-3x$, $-5x$, $-7x$, $-x$ is $-16x$.

Rule IV. *If the terms are not all of the same sign, add together separately the coefficients of all the positive terms and the coefficients of all the negative terms; the difference of these two results, preceded by the sign of the greater, will give the coefficient of the sum required.*

Example 1. The sum of $17x$ and $-8x$ is $9x$, for the difference of 17 and 8 is 9, and the greater is positive.

Example 2. To find the sum of $8a$, $-9a$, $-a$, $3a$, $4a$, $-11a$, a.

The sum of the coefficients of the positive terms is 16.

.... negative .... 21.

The difference of these is 5, and the sign of the greater is negative; hence the required sum is $-5a$.

We need not however adhere strictly to this rule, for the terms may be added or subtracted in the order we find most convenient.

This process is called **collecting terms.**

21. When quantities are connected by the signs + and −, the resulting expression is called their **algebraical sum.**

Thus $11a-27a+13a=-3a$ states that the algebraical sum of $11a$, $-27a$, $13a$ is equal to $-3a$.

22. The sum of two quantities numerically equal but with opposite signs is zero. Thus the sum of $5a$ and $-5a$ is 0.

EXAMPLES II.

Find the sum of

1. $2a$, $3a$, $6a$, a, $4a$.
2. $4x$, x, $5x$, $6x$, $8x$.
3. $6b$, $11b$, $8b$, $9b$, $5b$.
4. $6c$, $7c$, $3c$, $16c$, $18c$, $101c$.
5. $2p$, p, $4p$, $7p$, $6p$, $12p$.
6. d, $9d$, $3d$, $7d$, $4d$, $6d$, $10d$.
7. $-2x$, $-6x$, $-10x$, $-8x$.
8. $-3b$, $-13b$, $-19b$, $-5b$.
9. $-y$, $-4y$, $-2y$, $-6y$, $-4y$.
10. $-17c$, $-34c$, $-9c$, $-6c$.
11. $-21y$, $-5y$, $-3y$, $-18y$.
12. $-4m$, $-13m$, $-17m$, $-59m$.
13. $-4s$, $3s$, s, $2s$, $-2s$, $-s$.
14. $11y$, $-9y$, $-7y$, $5y$, $7y$.
15. $3x$, $-10x$, $-7x$, $12x$, $2x$.
16. $8ab$, $-6ab$, $5ab$, $-4ab$.
17. $2xy$, $-4xy$, $-3xy$, xy, $7xy$.
18. $5pq$, $-8pq$, $8pq$, $-4pq$.
19. abc, $-3abc$, $2abc$, $-5abc$.
20. $-xyz$, $-2xyz$, $7xyz$, $-xyz$.

Find the value of

21. $-9a^2+11a^2+3a^2-4a^2$.
22. $3b^3-2b^3+7b^3-9b^3$.
23. $11a^3+3a^3-8a^3-7a^3+2a^3$.
24. $2x^3-3x^3-6x^3-9x^3$.
25. $a^2b^2-7a^2b^2+8a^2b^2+9a^2b^2$.
26. $a^2x-11a^2x+3a^2x-2a^2x$.
27. $2p^3q^2-31p^3q^2+17p^3q^2$.
28. $7m^4n-15m^4n+3m^4n$.
29. $9abcd-11abcd-41abcd$.
30. $13pqx-5xpq-19qpx$.

CHAPTER III.

SIMPLE BRACKETS. ADDITION.

23. WHEN a number of arithmetical quantities are connected together by the signs + and −, the value of the result is the same in whatever order the terms are taken. This also holds in the case of algebraical quantities.

Thus $a-b+c$ is equivalent to $a+c-b$, for in the first of the two expressions b is taken from a, and c added to the result; in the second c is added to a, and b taken from the result. Similar reasoning applies to all algebraical expressions. Hence we may write the terms of an expression in any order we please.

Thus it appears that the expression $a-b$ may be written in the equivalent form $-b+a$.

To illustrate this we may suppose, as in Art. 18, that a represents a gain of a dollars, and $-b$ a loss of b dollars: it is clearly immaterial whether the gain precedes the loss, or the loss precedes the gain.

24. Brackets ( ) are used to indicate that the terms enclosed within them are to be considered as one quantity. The full use of brackets will be considered in Chap. VII.; here we shall deal only with the simpler cases.

$8+(13+5)$ means that 13 and 5 are to be added and their sum added to 8. It is clear that 13 and 5 may be added separately or together without altering the result.

Thus $$8+(13+5)=8+13+5=26.$$

Similarly $a+(b+c)$ means that the sum of b and c is to be added to a.

Thus $$a+(b+c)=a+b+c.$$

$8+(13-5)$ means that to 8 we are to add the excess of 13 over 5; now if we add 13 to 8 we have added 5 too much, and must therefore take 5 from the result.

Thus $$8+(13-5)=8+13-5=16.$$

Similarly $a+(b-c)$ means that to a we are to add b, diminished by c.

Thus $$a+(b-c)=a+b-c.$$

In like manner,

$$a+b-c+(d-e-f)=a+b-c+d-e-f.$$

By considering these results we are led to the following rule:

Rule. *When an expression within brackets is preceded by the sign +, the brackets can be removed without making any change in the expression.*

25. The expression $a-(b+c)$ means that from a we are to take the sum of b and c. The result will be the same whether b and c are subtracted separately or in one sum. Thus

$$a-(b+c)=a-b-c.$$

Again, $a-(b-c)$ means that from a we are to subtract the excess of b over c. If from a we take b we get $a-b$; but by so doing we shall have taken away c too much, and must therefore add c to $a-b$. Thus

$$a-(b-c)=a-b+c.$$

In like manner,

$$a-b-(c-d-e)=a-b-c+d+e.$$

Accordingly the following rule may be enunciated:

Rule. *When an expression within brackets is preceded by the sign −, the brackets may be removed if the sign of every term within the brackets be changed.*

Addition of Unlike Terms.

26. When two or more *like* terms are to be added together we have seen that they may be collected and the result expressed as a *single* like term. If, however, the terms are *unlike* they cannot be collected; thus in finding the sum of two unlike quantities a and b, all that can be done is to connect them by the sign of addition and leave the result in the form $a+b$.

27. We have now to consider the meaning of an expression like $a+(-b)$. Here we have to find the result of taking a negative quantity $-b$ together with a positive quantity a. Now $-b$ implies a decrease, and to add it to a is the same in effect as to subtract b; thus

$$a+(-b)=a-b;$$

that is, the algebraical sum of a and $-b$ is expressed by $a-b$.

28. It will be observed that in Algebra the word *sum* is used in a wider sense than in Arithmetic. Thus, in the language of Arithmetic, $a-b$ signifies that b is to be subtracted from a,

and bears that meaning only; but in Algebra it is also taken to mean the sum of the two quantities a and $-b$ without any regard to the relative magnitudes of a and b.

Example 1. Find the sum of $3a-5b+2c$, $2a+3b-d$, $-4a+2b$.

$$\begin{aligned}\text{The sum} &= (3a-5b+2c)+(2a+3b-d)+(-4a+2b)\\ &= 3a-5b+2c+2a+3b-d-4a+2b\\ &= 3a+2a-4a-5b+3b+2b+2c-d\\ &= a+2c-d,\end{aligned}$$

by collecting like terms.

The addition is however more conveniently effected by the following rule:

Rule. *Arrange the expressions in lines so that the like terms may be in the same vertical columns: then add each column beginning with that on the left.*

$$\begin{array}{rrrr} 3a & -5b & +2c & \\ 2a & +3b & & -d \\ -4a & +2b & & \\ \hline a & & +2c & -d \end{array}$$

The algebraical sum of the terms in the first column is a, that of the terms in the second column is zero. The single terms in the third and fourth columns are brought down without change.

Example 2. Add together $-5ab+6bc-7ac$; $8ab+3ac-2ad$; $-2ab+4ac+5ad$; $bc-3ab+4ad$.

$$\begin{array}{rrrr} -5ab & +6bc & -7ac & \\ 8ab & & +3ac & -2ad \\ -2ab & & +4ac & +5ad \\ -3ab & +bc & & +4ad \\ \hline -2ab & +7bc & & +7ad \end{array}$$

Here we first rearrange the expressions so that like terms are in the same vertical columns, and then add up each column separately.

EXAMPLES III. a.

Find the sum of

1. $3a+2b-5c$; $-4a+b-7c$; $4a-3b+6c$.
2. $3x+2y+6z$; $x-3y-3z$; $2x+y-3z$.
3. $4p+3q+5r$; $-2p+3q-8r$; $p-q+r$.
4. $7a-5b+3c$; $11a+2b-c$; $16a+5b-2c$.
5. $8l-2m+5n$; $-6l+7m+4n$; $-l-4m-8n$.
6. $5a-7b+3c-4d$; $6b-5c+3d$; $b+2c-d$.
7. $2a+4b-5x$; $2b-5x$; $-3a+2y$; $-6b+8x+y$.
8. $7x-5y-7z$; $4x+y$; $5z$; $5x-3y+2z$.

9. $a-2b+7c+3$; $2b-3c+5$; $3c+2a$; $a-8-7c$.
10. $5-x-y$; $7+2x$; $3y-2$; $-4+x-2y$.
11. $25a-15b+c$; $4c-10b+13a$; $a-c+20b$.
12. $2a-3b-2c+2x$; $5x+3b-7c$; $9c-6x-2a$.
13. $3a-5c+2b-2d$; $b+2d-a$; $5c+3f+3e-2a-3b$.
14. $p-q+7r$; $6q+r-p$; $q-3p-r$; $6q-7p$.
15. $17ab-13kl-5xy$; $7xy$; $12kl-5ab$; $3xy-4kl-ab$.
16. $2ax-3by-2cz$; $2by-ax+7cz$; $ax-4cz+7by$; $cz-6by$.
17. $3ax+cz-4by$; $7by-8ax-cz$; $-3by+9ax$.
18. $3+5cd$; $2fg-3st$; $1-5cd$; $-4+2st-fg$.
19. $5cx+3fy-2+2s$; $-2fy+6-9s$; $-3s-4+2cx-fy$.
20. $-3ab+7cd-5qr$; $2ry+8qr-cd$; $2cd-3qr+ab-2ry$.

29. Different powers of the same letter are **unlike terms**; thus the result of adding together $2x^3$ and $3x^2$ cannot be expressed by a single term, but must be left in the form $2x^3+3x^2$.

Similarly the algebraical sum of $5a^2b^2$, $-3ab^3$, and $-b^4$ is $5a^2b^2-3ab^3-b^4$. This expression is in its simplest form and cannot be abridged.

Example. Find the sum of $6x^3-5x$, $2x^2$, $5x$, $-2x^3$, $-3x^2$, 2.

$$\begin{aligned} \text{The sum} &= 6x^3-5x+2x^2+5x-2x^3-3x^2+2 \\ &= 6x^3-2x^3+2x^2-3x^2-5x+5x+2 \\ &= 4x^3-x^2+2. \end{aligned}$$

This result is in *descending* powers of x.

30. In adding together several algebraical expressions containing terms with different powers of the same letter, it will be found convenient to arrange all expressions in *descending* or *ascending* powers of that letter. This will be made clear by the following example.

Example 1. Add together $3x^3+7+6x-5x^2$; $2x^2-8-9x$; $4x-2x^3+3x^2$; $3x^3-9x-x^2$; $x-x^2-x^3+4$.

$$\begin{array}{rrrr} 3x^3 & -5x^2 & +6x & +7 \\ & 2x^2 & -9x & -8 \\ -2x^3 & +3x^2 & +4x & \\ 3x^3 & -x^2 & -9x & \\ -x^3 & -x^2 & +x & +4 \\ \hline 3x^3 & -2x^2 & -7x & +3 \end{array}$$

In writing down the first expression we put in the first term the highest power of x, in the second term the next highest power, and so on till the last term in which x does not appear. The other expressions are arranged in the same way, so that in each column we have *like powers of the same letter*.

Example 2. Add together

$$3ab^2-2b^3+a^3;\quad 5a^2b-ab^2-3a^3;\quad 8a^3+5b^3;\quad 9a^2b-2a^3+ab^2.$$

$$\begin{array}{rrrr}
-2b^3 & +3ab^2 & & +a^3 \\
& -ab^2 & +5a^2b & -3a^3 \\
5b^3 & & & +8a^3 \\
& ab^2 & +9a^2b & -2a^3 \\
\hline
3b^3 & +3ab^2 & +14a^2b & +4a^3
\end{array}$$

Here each expression contains powers of two letters, and is arranged according to *descending* powers of b, and *ascending* powers of a.

EXAMPLES III. b.

Find the sum of the following expressions:

1. $x^2+3xy-3y^2$; $-3x^2+xy+2y^2$; $2x^2-3xy+y^2$.
2. $2x^2-2x+3$; $-2x^2+5x+4$; x^2-2x-6.
3. $5x^3-x^2+x-1$; $2x^2-2x+5$; $-5x^3+5x-4$.
4. $a^3-a^2b+5ab^2+b^3$; $-a^3-10ab^2+b^3$; $2a^2b+5ab^2-b^3$.
5. $3x^3-9x^2-11x+7$; $2x^3-5x^2+2$; $5x^3+15x^2-7x$; $8x-9$.
6. x^5-5x^3+8x; $7x^5+4x^3+5x$; $8x^3-9x$; $2x^5-7x^3-4x$.
7. $4m^3+2m^2-5m+7$; $3m^3+6m^2-2$; $-5m^2+3m$; $2m-6$.
8. ax^3-4bx^2+cx; $3bx^2-2cx-d$; bx^2+2d; $2ax^3+d$.
9. $py^2-9qy+7r$; $-2py^2+3qy-6r$; $7qy-4r$; $3py^2$.
10. $5y^3+20y^2+3y-1$; $-2y+5-7y^2$; $-3y^2-4+2y^3-y$.
11. $2-a+8a^2-a^3$; $2a^3-3a^2+2a-2$; $-3a+7a^3-5a^2$.
12. $1+2y-3y^2-5y^3$; $-1+2y^2-y$; $5y^3+3y^2+4$.
13. $a^2x^3-3a^3x^2+x$; $5x+7a^3x^2$; $4a^3x^2-a^2x^3-5x$.
14. $x^5-4x^4y-5x^3y^3$; $3x^4y+2x^3y^3-6xy^4$; $3x^3y^3+6xy^4-y^5$.
15. $a^3-4a^2b+6abc$; $a^2b-10abc+c^3$; b^3+3a^2b+abc.
16. ap^5-6bp^3+7cp; $5-6cp+5bp^3$; $3-2ap^5$; $2cp-7$.
17. $c^7-2c^5+11c^6$; $-2c^7-3c^6+5c^5$; $4c^6-10c^5$; $4c^7-c^6$.
18. $4h^3-7+3h^4-2h$; $7h-3h^3+2-h^4$; $2h^4+2h^3-5$.
19. $3x^3+2y^2-5x+2$; $7x^3-5y^2+7x-5$; $9x^3+11-8x+4y^2$; $6x-y^2-18x^3-7$.
20. $x^2+2xy+3y^2$; $3z^2+2yz+y^2$; x^2+3z^2+2xz; $z^2-3xy-3yz$; $xy+xz+yz-6z^2-4y^2-2x^2$.

CHAPTER IV.

SUBTRACTION.

31. THE simplest cases of Subtraction have already come under the head of addition of *like* terms, of which some are negative. [Art. 20.]

Thus
$$5a-3a=\ \ 2a,$$
$$3a-7a=-4a,$$
$$-3a-6a=-9a.$$

Since subtraction is the reverse of addition,
$$+b-b=0;$$
$$\therefore\ a=a+b-b.$$

Now subtract $-b$ from the left-hand side and erase $-b$ on the right; we thus get
$$a-(-b)=a+b.$$

This also follows directly from the rule for removing brackets. [Art. 25.]

Thus
$$3a-(-5a)=3a+5a$$
$$=8a,$$
and
$$-3a-(-5a)=-3a+5a$$
$$=2a.$$

Subtraction of Unlike Terms.

32. We may proceed as in the following example.

Example. Subtract $3a-2b-c$ from $4a-3b+5c$.

The difference

$$=4a-3b+5c-(3a-2b-c)$$
$$=4a-3b+5c-3a+2b+c$$
$$=4a-3a-3b+2b+5c+c$$
$$=a-b+6c.$$

The expression to be subtracted is first enclosed in brackets with a minus sign prefixed, then on removal of the brackets the like terms are combined by the rules already exclaimed in Art. 20.

It is, however, more convenient to arrange the work as follows, the signs of all the terms in the lower line being changed.

	$4a-3b+5c$	The like terms are written in the same vertical column, and each column is treated separately.
	$-3a+2b+c$	
by *addition*,	$a-\ b+6c$	

Rule. *Change the sign of every term in the expression to be subtracted, and add to the other expression.*

Note. It is not necessary that in the expression to be subtracted the signs should be *actually* changed; the operation of changing signs ought to be performed mentally.

Example 1. From $5x^2+xy$ take $2x^2+8xy-7y^2$.

$5x^2+\ xy$	In the first column we combine mentally $5x^2$ and $-2x^2$, the algebraic sum of which is $3x^2$. In the last column the sign of the term $-7y^2$ has to be changed before it is put down in the result.
$2x^2+8xy-7y^2$	
$3x^2-7xy+7y^2$	

Example 2. Subtract $3x^2-2x$ from $1-x^3$.

Terms containing different powers of the same letter being *unlike* must stand in different columns.

$-x^3\qquad\qquad +1$	In the first and last columns, as there is nothing to be subtracted, the terms are put down without change of sign. In the second and third columns each sign has to be changed.
$\quad 3x^2-2x$	
$-x^3-3x^2+2x+1$	

The re-arrangement of terms in the first line is not *necessary*, but it is convenient, because it gives the result of subtraction in descending powers of x.

EXAMPLES IV.

Subtract

1. $a+2b-c$ from $2a+3b+c$.
2. $2a-b+c$ from $3a-5b-c$.
3. $3x+y-z$ from $x-4y+3z$.
4. $x+8y+8z$ from $10x-7y-6z$.
5. $-m-3n+p$ from $-2m+n-3p$.
6. $3p-2q+r$ from $4p-7q+3r$.
7. $a-7b-3c$ from $-4a+3b+8c$.
8. $-a-b-9c$ from $-a+b-9c$.

Subtract

9. $3x-5y-7z$ from $2x+3y-4z$.
10. $-4x-2y+11z$ from $-x+2y-13z$.
11. $-2x-5y$ from $x+3y-2z$.
12. $3x-y-8z$ from $x+2y$.
13. $m-2n-p$ from $m+2n$.
14. $2p-3q-r$ from $2q-4r$.
15. $ab-2cd-ac$ from $-ab-3cd+2ac$.
16. $3ab+6cd-3ac-5bd$ from $3ab+5cd-4ac-6bd$.
17. $-xy+yz-zx$ from $2xy+zx$.
18. $-2pq-3qr+4rs$ from $qr-4rs$.
19. $-mn+11np-3pm$ from $-11np$.
20. x^2y-2xy^2+3xyz from $2x^2y+3xy^2-xyz$.

From

21. x^3-3x^2+x take $-x^3+3x^2-x$.
22. $-2x^3-x^2$ take x^3-x^2-x.
23. a^3+b^3-3abc take b^3-2abc.
24. $-8+6bc+b^2c^2$ take $4-3bc-5b^2c^2$.
25. $3p^3-2p^2q+7pq^2$ take $p^2q-3pq^2+q^3$.
26. $7+x-x^2$ take $5-x+x^2+x^3$.
27. $-4+x^2y-xyz$ take $-3-2x^2y+11xyz$.
28. $-8a^2x^2+5x^2+15$ take $9a^2x^2-8x^2-5$.
29. p^3+r^3-3pqr take r^3+q^3+3pqr.
30. $1-3x^2$ take x^3-3x^2+1.
31. $2+3x-7x^2$ take $3x^2-3x-2$.
32. x^3+11x^2+4 take $8x^2-5x-3$.
33. a^3+5-2a^2 take $8a^3+3a^2-7$.
34. $x^4+3x^3-x^2-8$ take $2x^4+3x^2-x+2$.
35. $1-2x+3x^2$ take $7x^3-4x^2+3x+1$.
36. x^2yz+y^2zx take $-3y^2zx-2xyz^2-x^2yz$.
37. $4a^3x^2-3ax^4+a^5$ take $3a^3x^2+7a^2x^3-a^5$.
38. $1-x+x^5-x^4-x^3$ take $x^4-1+x-x^2$.
39. $-8mn^2+15m^2n+n^3$ take $m^3-n^3+8mn^2-7m^2n$.
40. $1-p^3$ take $2p^3-3pq^2-2q^3$.

33. The following exercise contains miscellaneous examples on the foregoing rules.

MISCELLANEOUS EXAMPLES I.

1. When $x=2$, $y=3$, $z=4$, find the value of the sum of $5x^2$, $-3xy$, z^2. Also find the value of $3z^x+3x^y$.

2. Add together $3ab+bc-ca$, $-ab+ca$, $ab-2bc+5ca$. From the sum take $5ca+bc-ab$.

3. Subtract the sum of $x-y+3z$ and $-2y-2z$ from the sum of $2x-5y-3z$ and $-3x+y+4z$.

4. Simplify (1) $3b-2b^2-(2b-3b^2)$.

(2) $3a-2b-(2b+a)-(a-5b)$.

5. Subtract $8c^2+8c-2$ from c^3-1.

---

6. When $x=3$, $a=2$, $y=4$, $z=0$, find the value of

(1) $2x^2-3ay+4xz^3$. (2) $\dfrac{5a^3x}{3y}$.

7. Add together $3a^2-7a+5$ and $2a^3+5a-3$, and diminish the result by $3a^2+2$.

8. Subtract $2b^2-2$ from $-2b+6$, and increase the result by $3b-7$.

9. Find the sum of $3x^2-4x+8$, $2x-3-x^2$, and $2x^2-2$, and subtract the result from $6x^2+3$.

10. What expression must be added to $5a^2-3a+12$ to produce $9a^2-7$?

---

11. Find the sum of $2x$, $-x^3$, $3x^2$, 2, $-5x$, -4, $3x^3$, $-5x^2$, 8; arrange the result in ascending powers of x.

12. From what expression must the sum of $5a^2-2$, $3a+a^2$, and $7-2a$ be subtracted to produce $3a^2+a-5$?

13. When $x=6$, find the numerical value of the sum of $1-x+x^2$, $2x^2-1$, and $x-x^2$.

14. Find the value of $6ax+(2by-cz)-(2ax-3by+4cz)-(cz+ax)$, when $a=0$, $b=1$, $c=2$, $x=8$, $y=3$, $z=4$.

15. Subtract the sum of x^3-3x^2, $2x^2-7x$, $8x-2$, $5-3x^3$, $2x^3-7$ from x^3+x^2+x+1.

16. What expression must be taken from the sum of p^4-3p^3, $2p+8$, $2p^2$, $2p^3-3p^4$, in order to produce $4p^4-3$?

17. What is the result when $-3x^3+2x^2-11x+5$ is subtracted from zero.

18. By how much does $b+c$ exceed $b-c$?

19. Find the algebraic sum of three times the square of x, twice the cube of x, $-x^3+x-2x^2$, and x^3-x-x^2+1.

20. Take p^2-q^2 from $3pq-4q^2$, and add the remainder to the sum of $4pq-p^2-3q^2$ and $2p^2+6q^2$.

---

21. Subtract $3b^3+2b^2-8$ from zero, and add the result to b^4-2b^3+3b.

22. By how much does the sum of $-m^3+2m-1$, m^2-3m, $2m^3-2m^2+5$, $3m^3+4m^2+5m+3$, fall short of $11m^3-8m^2+3m$?

23. Find the sum of $8x^5-4x^3y^2$, $7x^4y-xy^4$, $3x^3y^2+2x^2y^3+5xy^4$, $y^5-4xy^4+x^3y^2$, $x^5-y^5+x^3y^2+xy^4-x^2y^3+3x^4y$, and arrange the result in descending powers of x.

24. To what expression must $3x-4x^3+7x^2+4$ be added so as to make zero? Give the answer in ascending powers of x.

25. Subtract $7x^2-3x-6$ from unity, and $x-5x^2$ from zero, and add the results.

26. When $a=4$, $b=3$, $c=2$, $d=0$, find the value of

(1) $3a^2-2bc-ad+3b^2cd$. (2) $\dfrac{2b^2c}{9a}$.

27. Find the sum of a, $-3a^2$, $4a$, $-5a$, 7, $-18a$, $4a^2$, -6, and arrange the result in descending powers of a.

28. Add together $4+3x^2+x^3$, x^3-x^2-11, x^3-2x^2+7, and subtract $2x^3+x^2-7$ from the result.

29. If $a=5x-3y+z$, $b=-2x+y-3z$, $c=x-5y+6z$, find the value of $a+b-c$.

30. If $x=2a^2-5a+3$, $y=-3a^2+a+8$, $z=5a^2-6a-5$, find the value of $x-(y+z)$.

CHAPTER V.

MULTIPLICATION.

34. MULTIPLICATION in its primary sense signifies repeated addition.

Thus $$3 \times 5 = 3 \text{ taken } 5 \text{ times}$$
$$= 3+3+3+3+3.$$
Here the multiplier contains 5 units, and the number of times we take 3 is the same as the number of units in 5.

Again $$a \times b = a \text{ taken } b \text{ times}$$
$$= a+a+a+a+\ldots, \text{ the number of terms being } b.$$
Also $3 \times 5 = 5 \times 3$; and so long as a and b denote positive whole numbers it is easy to shew that
$$a \times b = b \times a.$$
Hence $$abc = a \times b \times c = (a \times b) \times c = b \times a \times c = bac$$
$$= b \times (a \times c) = b \times c \times a = bca.$$

Similarly we can show that the product of three positive integral quantities a, b, c is the same in whatever order the factors are written.

Example. $2a \times 3b = 2 \times a \times 3 \times b = 2 \times 3 \times a \times b = 6ab.$

35. When the quantities to be multiplied together are not positive whole numbers, the definition of multiplication has to be modified. For example to multiply 3 by $\frac{4}{7}$, we perform on 3 that operation which when performed on unity gives $\frac{4}{7}$; that is, we must divide 3 into 7 equal parts and take 4 of them.

By taking multiplication in this sense, the statement $ab = ba$ can be extended so as to include every case in which a and b stand for positive quantities.

It follows as in the previous article that the product of a number of positive factors is the same in whatever order the factors are written.

36. Since, by definition, $a^3 = aaa$, and $a^5 = aaaaa$;
$$\therefore \; a^3 \times a^5 = aaa \times aaaaa = aaaaaaaa = a^8 = a^{3+5};$$
that is, the index of a in the product is the sum of the indices of a in the factors of the product.

Again, $$5a^2 = 5aa, \text{ and } 7a^3 = 7aaa;$$
$$\therefore \; 5a^2 \times 7a^3 = 5 \times 7 \times aaaaa = 35a^5.$$

When the expressions to be multiplied together contain powers of different letters, a similar method is used.

Example. $5a^3b^2 \times 8a^2bx^3 = 5aaabb \times 8aabxxx$
$= 40a^5b^3x^3.$

Note. The beginner must be careful to observe that in this process of multiplication *the indices of one letter cannot combine in any way with those of another.* Thus the expression $40a^5b^3x^3$ admits of no further simplification.

37. **Rule.** *To multiply two simple expressions together, multiply the coefficients together and prefix their product to the product of the different letters, giving to each letter an index equal to the sum of the indices that letter has in the separate factors.*

The rule may be extended to cases where more than two expressions are to be multiplied together.

Example 1. Find the product of x^2, x^3, and x^8.

The product $= x^2 \times x^3 \times x^8 = x^{2+3} \times x^8 = x^{2+3+8} = x^{13}$.

The product of three or more expressions is called **the continued product.**

Example 2. Find the continued product of $5x^2y^3$, $8y^2z^5$, and $3xz^4$.

The product $= 5x^2y^3 \times 8y^2z^5 \times 3xz^4 = 120x^3y^5z^9$.

38. By definition, $(a+b)\,m = m+m+m+ \dots$ taken $a+b$ times
$=(m+m+m+\dots$ taken a times$)$,
together with $(m+m+m+\dots$ taken b times$)$
$=am+bm.$

Also $(a-b)\,m = m+m+m+ \dots$ taken $a-b$ times
$=(m+m+m+\dots$ taken a times$)$,
diminished by $(m+m+m+\dots$ taken b times$)$
$=am-bm.$

Similarly $(a-b+c)\,m = am-bm+cm.$

Thus it appears that *the product of a compound expression by a single factor is the algebraic sum of the partial products of each term of the compound expression by that factor.*

Examples. $3(2a+3b-4c) = 6a+9b-12c.$

$(4x^2-7y-8z^3) \times 3xy^2 = 12x^3y^2 - 21xy^3 - 24xy^2z^3.$

EXAMPLES V. a.

Find the value of

1. $5x \times 7$. 2. $3 \times 2b$. 3. $x^2 \times x^3$. 4. $5x \times 6x^2$.

5. $6c^3 \times 7c^4$. 6. $9y^2 \times 5y^5$. 7. $3m^3 \times 5m^5$. 8. $4a^6 \times 6a^4$.

9. $3x \times 4y$. 10. $5a \times 6b^2$. 11. $4c^2 \times 5d^5$. 12. $3p^4 \times 5q^5$.

13. $6ax \times 5ax$. 14. $3qr \times 4qr$. 15. $ab \times ab$. 16. $3ac \times 5ad$.

17. $a^3x \times a^4x^3$. 18. $3x^3y^2 \times 4y^5$. 19. $a^3b^5 \times a^5b^4$. 20. $a^4 \times 3a^5b^3$.

21. $a^2 \times a^3b \times 5ab^4$. 22. $pr^4 \times 6p^3r \times 7pr^5$.

23. $6x^3y \times xy \times 9x^4y^2$. 24. $7a^2 \times 3b^3 \times 5c^4$.

25. $6xy^2 \times 7yz^2 \times xz^3$. 26. $3abcd \times 5bca^2 \times 4cabd$.

Multiply

27. $ab - ac$ by a^2c. 28. $x^2y - x^3z + 4yz^5$ by x^3yz^2.

29. $5a^2 - 3b^2$ by $3ab^2c^4$. 30. $a^2b - 5ab + 6a$ by $3a^3b$.

31. $a^2 - 2b^3$ by $3x^2$. 32. $2ax^2 - b^3y + 3$ by a^2xy.

33. $7p^2q - pq^2 + 1$ by $2p^2$. 34. $m^2 + 5mn - 3n^2$ by $4m^2n$.

35. $xy^2 - 3x^2z - 2$ by $3yz$. 36. $a^3 - 3a^2x$ by $2a^2bx$.

39. Since $\qquad (a-b)m = am - bm,$ $\qquad$ [Art. 38.]

by putting $c - d$ in the place of m, we have

$$\begin{aligned}(a-b)(c-d) &= a(c-d) - b(c-d)\\ &= (c-d)a - (c-d)b\\ &= (ac - ad) - (bc - bd)\\ &= ac - ad - bc + bd.\end{aligned}$$

If we consider each term on the right-hand side of this result, and the way in which it arises, we find that

$$\begin{aligned}(+a) \times (+c) &= +ac.\\ (-b) \times (-d) &= +bd.\\ (-b) \times (+c) &= -bc.\\ (+a) \times (-d) &= -ad.\end{aligned}$$

These results enable us to state what is known as the **Rule of Signs** in multiplication.

Rule of Signs. *The product of two terms with like signs is positive; the product of two terms with unlike signs is negative.*

40. The rule of signs, and especially the use of the negative *multiplier, will probably present some difficulty to the beginner. Perhaps the following numerical instances may be useful in illustrating* the interpretation that may be given to multiplication by a negative quantity.

To multiply 3 by -4 we must do to 3 what is done to unity to obtain -4. Now -4 means that unity is taken 4 times and the result made negative: therefore $3\times(-4)$ implies that 3 is to be taken 4 times and the product made negative.

But 3 taken 4 times gives $+12$;

$$\therefore\ 3\times(-4)=-12.$$

Similarly -3×-4 indicates that -3 is to be taken 4 times, and the sign changed; the first operation gives -12, and the second $+12$.

Thus $$(-3)\times(-4)=+12.$$

Hence, *multiplication by a negative quantity indicates that we are to proceed just as if the multiplier were positive, and then change the sign of the product.*

Example 1. Multiply $4a$ by $-3b$.

By the rule of signs the product is negative; also $4a\times 3b=12ab$;

$$\therefore\ 4a\times(-3b)=-12ab.$$

Example 2. Multiply $-5ab^3x$ by $-ab^3x$.

Here the absolute value of the product is $5a^2b^6x^2$, and by the rule of signs the product is positive;

$$\therefore\ (-5ab^3x)\times(-ab^3x)=5a^2b^6x^2.$$

Example 3. Find the continued product of $3a^2b$, $-2a^3b^2$, $-ab^4$.

$3a^2b\times(-2a^3b^2)=-6a^5b^3$;
$(-6a^5b^3)\times(-ab^4)=+6a^6b^7$.

Thus the complete product is $6a^6b^7$.

This result, however, may be written down at once: for

$$3a^2b\times 2a^3b^2\times ab^4=6a^6b^7,$$

and by the rule of signs the required product is positive.

Example 4. Multiply $6a^3-5a^2b-4ab^2$ by $-3ab^2$.

The product is the algebraical sum of the partial products formed according to the rule enunciated in Art. 37;

thus $$(6a^3-5a^2b-4ab^2)\times(-3ab^2)=-18a^4b^2+15a^3b^3+12a^2b^4.$$

EXAMPLES V. b.

Multiply together

1. $a, -2$.
2. $-3, 4x$.
3. $-x^2, -x^3$.
4. $-5m, 3m^3$.
5. $-4q, 3q^2$.
6. $-4y^3, -4y^3$.
7. $-3m^3, 3m^3$.
8. $4x^4, -4x^4$.
9. $-3x, -4y$.
10. $-5a^2, 4x$.
11. $-3p^2, -4q^5$.
12. $3ab, -4ab$.
13. $3a^2, -b^2, 2ab$.
14. $-a, -b, -c^2$.
15. $3a^2, -2b, -4c^3, -d$.
16. $-3ab, -4ac, 3bc$.
17. $-2a^3, -3a^2b, -6$.
18. $-2p, -3q, 4s, -t$.

Multiply

19. $-ab+ac-bc$ by $-ab$.
20. $-3a^2-4ax+5x^2$ by $-a^2x^3$.
21. $a^2c-ac^3+c^4$ by $-a^3c$.
22. $-2ab+cd-ef$ by $-3x^2y^2$.

41. To further illustrate the use of the rule of signs, we add a few examples in substitution where some of the symbols denote negative quantities.

Example 1. If $a = -4$, find the value of a^3.

Here $$a^3 = (-4)^3 = (-4)\times(-4)\times(-4) = -64.$$

By repeated applications of the rule of signs it may easily be shown that any *odd* power of a negative quantity is *negative*, and any *even* power of a negative quantity is *positive*.

Example 2. If $a = -1$, $b = 3$, $c = -2$, find the value of $-3a^4bc^3$.

Here $$\begin{aligned} -3a^4bc^3 &= -3\times(-1)^4\times3\times(-2)^3 \\ &= -3\times(+1)\times3\times(-8) \\ &= 72. \end{aligned}$$

We write down at once, $(-1)^4 = +1$, and $(-2)^3 = -8$.

EXAMPLES V. c.

If $a = -1$, $b = 0$, $c = -2$, $n = 1$, $q = -3$, find the value of

1. $3c$.
2. $-5a$.
3. an.
4. $(-a)^2$.
5. $-3c^2$.
6. $(-q)^4$.
7. $-2a^3$.
8. $-ac$.
9. ab.
10. $-acn$.
11. $-3a^4$.
12. $4(-c)^3$
13. $2abc^2$.
14. $-c^2$.
15. $-(a)^4$.
16. $-3a^2q$.
17. $-a^3n^2$.
18. ac^3.
19. $-a^3c^2$.
20. c^3q^2.

If $a=-3$, $c=1$, $k=0$, $x=5$, $y=-1$, find the value of

21. $3a-2y+4k$.
22. $-4c-3x+2y$.
23. $-4a+5y-x$.
24. $ac-3cy-yk$.
25. $2ay-kx+4k^2$.
26. $a^2-2c^2+3y^2$.
27. $-a^2-ay+3y^2$.
28. $ax-yx-cy$.
29. $c^2-y^2-c^3+y^3$.
30. a^3-x^2-2y.
31. $c^2y^2-2ac^2+ck^2$.
32. $acy-y^4+2a^2$.

Multiplication of Compound Expressions.

42. *To find the product of $a+b$ and $c+d$.*

From Art. 38, $(a+b)m=am+bm$;

replacing m by $c+d$, we have

$$\begin{aligned}(a+b)(c+d)&=a(c+d)+b(c+d)\\&=(c+d)a+(c+d)b\\&=ac+ad+bc+bd.\end{aligned}$$

Similarly it may be shown that

$$\begin{aligned}(a-b)(c+d)&=ac+ad-bc-bd;\\(a+b)(c-d)&=ac-ad+bc-bd;\\(a-b)(c-d)&=ac-ad-bc+bd.\end{aligned}$$

43. When one or both of the expressions to be multiplied together contain more than two terms a similar method may be used. For instance

$$(a-b+c)m=am-bm+cm;$$

replacing m by $x-y$, we have

$$\begin{aligned}(a-b+c)(x-y)&=a(x-y)-b(x-y)+c(x-y)\\&=(ax-ay)-(bx-by)+(cx-cy)\\&=ax-ay-bx+by+cx-cy.\end{aligned}$$

44. The preceding results enable us to state the general rule for multiplying together any two compound expressions.

Rule. *Multiply each term of the first expression by each term of the second. When the terms multiplied together have like signs, prefix to the product the sign $+$, when unlike prefix $-$; the algebraical sum of the partial products so formed gives the complete product.*

45. It should be noticed that the product of $a+b$ and $x-y$ is briefly expressed by $(a+b)(x-y)$, in which the brackets indicate that the expression $a+b$ taken as a whole is to be multiplied by the expression $x-y$ taken as a whole. By the

above rule, the value of the product is the algebraical sum of the partial products $+ax$, $+bx$, $-ay$, $-by$; the sign of each product being determined by the rule of signs.

Example 1. Multiply $x+8$ by $x+7$.

The product

$$\begin{aligned} &= (x+8)(x+7) \\ &= x^2+8x+7x+56 \\ &= x^2+15x+56. \end{aligned}$$

The operation is more conveniently arranged as follows:

$$\begin{array}{rl} & x + 8 \\ & x + 7 \\ \hline & x^2 + 8x \\ & \quad + 7x + 56 \\ \hline \text{by addition,} & x^2 + 15x + 56. \end{array}$$

We begin on the left and work to the right, placing the second result one place to the right, so that like terms may stand in the same vertical column.

Example 2. Multiply $2x-3y$ by $4x-7y$.

$$\begin{array}{rl} & 2x - 3y \\ & 4x - 7y \\ \hline & 8x^2 - 12xy \\ & \quad - 14xy + 21y^2 \\ \hline \text{by addition,} & 8x^2 - 26xy + 21y^2. \end{array}$$

EXAMPLES V. d.

Find the product of

1. $a+7$, $a+5$.	2. $x-3$, $x+4$.	3. $a-6$, $a-7$.
4. $y-4$, $y+4$.	5. $x+9$, $x-8$.	6. $c-8$, $c+8$.
7. $k+5$, $k-5$.	8. $m-9$, $m+12$.	9. $x-12$, $x+11$.
10. $a-14$, $a+1$.	11. $p-10$, $p+10$.	12. $d+7$, $d+7$.
13. $x-4$, $-x+4$.	14. $-y+3$, $-y-3$.	15. $-a+4$, $-a+5$.
16. $x-10$, $-x+8$.	17. $-k+4$, $-k-7$.	18. $-y-7$, $-y-7$.
19. $2a-5$, $3a+2$.	20. $x-7$, $2x+5$.	21. $3x-4$, $2x+3$.
22. $3y-5$, $y+7$.	23. $5m-4$, $7m-3$.	24. $7p-2$, $2p+7$.
25. $x-3a$, $2x+3a$.	26. $3a-2b$, $2a+3b$.	27. $5c+4d$, $5c-4d$.
28. $a-2x$, $3a+2x$.	29. $7b+c$, $7b-2c$.	30. $2a-5c$, $2a+5c$.
31. $3x-5y$, $4x+y$.	32. $2y-3z$, $2y+3z$.	33. $xy+2b$, $xy-2b$.
34. $2x-3a$, $2x+3b$.	35. $3x-4y$, $2a+3b$.	36. $mn-p$, $2xy+3z$.

46. We shall now give a few examples of greater difficulty.

Example 1. Find the product of $3x^2-2x-5$ and $2x-5$.

$$\begin{array}{llll} 3x^2 & -2x & -5 & \\ 2x & -5 & & \\ \hline 6x^3 & -4x^2 & -10x & \\ & -15x^2 & +10x & +25 \\ \hline 6x^3 & -19x^2 & & +25. \end{array}$$

Each term of the first expression is multiplied by $2x$, the first term of the second expression; then each term of the first expression is multiplied by -5; like terms are placed in the same columns and the results added.

Example 2. Multiply $a-b+3c$ by $a+2b$.

$$\begin{array}{lllll} a & -b & +3c & & \\ a & +2b & & & \\ \hline a^2 & -ab & +3ac & & \\ & 2ab & & -2b^2 & +6bc \\ \hline a^2 & +ab & +3ac & -2b^2 & +6bc. \end{array}$$

47. If the expressions are not arranged according to powers, ascending or descending, of some common letter, a rearrangement will be found convenient.

Example. Find the product of $2a^2+4b^2-3ab$ and $3ab-5a^2+4b^2$.

$$\begin{array}{lllll} 2a^2 & -3ab & +4b^2 & & \\ -5a^2 & +3ab & +4b^2 & & \\ \hline -10a^4 & +15a^3b & -20a^2b^2 & & \\ & +6a^3b & -9a^2b^2 & +12ab^3 & \\ & & 8a^2b^2 & -12ab^3 & +16b^4 \\ \hline -10a^4 & +21a^3b & -21a^2b^2 & & +16b^4. \end{array}$$

The re-arrangement is not *necessary*, but convenient, because it makes the collection of like terms more easy.

EXAMPLES V. e.

Multiply together

1. $x^2-3x-2,\ 2x-1.$
2. $4a^2-a-2,\ 2a+3.$
3. $2y^2-3y+1,\ 3y-1.$
4. $3x^2+4x+5,\ 4x-5.$
5. $2a^2-3a-6,\ a-2.$
6. $5b^2-2b+3,\ -2b-3.$
7. $3x^2-2x+7,\ 2x-7.$
8. $5c^2-4c+3,\ -2c+1.$
9. $x^2+x-2,\ x^2-x+2.$
10. $x^2-2x+5,\ x^2-2x+5.$
11. $2a^2-3a-6,\ a^2-a+2.$
12. $2k^2-3k-1,\ 3k^2-k-1.$
13. $a+b-c,\ a-b+c.$
14. $a-2b-3c,\ a-2b+3c.$

15. $x^2-xy+y^2,\ x^2+xy+y^2.$ 16. $a^2-2ax+2x^2,\ a^2+2ax+2x^2.$
17. $a^2-b^2-3c^2,\ -a^2-b^2-3c^2.$ 18. $x^3-3x^2-x,\ x^2-3x+1.$
19. $a^3-6a+5,\ a^3+6a-5.$ 20. $2y^4-4y^2+1,\ 2y^4-4y^2-1.$
21. $5m^2+3-4m,\ 5-4m+3m^2.$ 22. $8a^3-2a^2-3a,\ 3a^2+1-5a.$
23. $2x+2x^3-3x^2,\ 3x+2+2x^2.$ 24. $a^3+b^3-a^2b^2,\ a^2b^2-a^3+b^3.$
25. $a^3+x^3+3ax^2+3a^2x,\ a^3+3ax^2-x^3-3a^2x.$
26. $5p^4-p^3+4p^2-2p+3,\ p^2-2p+3.$
27. $m^5-2m^4+3m^3-4m^2,\ 4m^6-3m^5+2m^4.$
28. $a^4+1+6a^2-4a^3-4a,\ a^3-1+3a-3a^2.$
29. $a^2+b^2+c^2+ab+ac-bc,\ a-b-c.$
30. $x^4+6x^2y^2+y^4-4x^3y-4xy^3,\ -x^4-y^4-6x^2y^2-4xy^3-4x^3y.$

48. Although the result of multiplying together two binomial factors, such as $x+8$ and $x-7$, can always be obtained by the methods already explained, it is of the utmost importance that the student should soon learn to write down the product rapidly *by inspection.*

This is done by observing in what way the coefficients of the terms in the product arise, and noticing that they result from the combination of the numerical coefficients in the two binomials which are multiplied together; thus

$$(x+8)(x+7)=x^2+8x+7x+56$$
$$=x^2+15x+56.$$
$$(x-8)(x-7)=x^2-8x-7x+56$$
$$=x^2-15x+56.$$
$$(x+8)(x-7)=x^2+8x-7x-56$$
$$=x^2+x-56.$$
$$(x-8)(x+7)=x^2-8x+7x-56$$
$$=x^2-x-56.$$

In each of these results we notice that:

1. The product consists of three terms.

2. The first term is the product of the first terms of the two binomial expressions.

3. The third term is the product of the second terms of the two binomial expressions.

4. The middle term has for its coefficient the sum of the numerical quantities (taken with their proper signs) in the second terms of the two binomial expressions.

The intermediate step in the work may be omitted, and the products written down at once, as in the following examples:

$$(x+2)(x+3)=x^2+5x+6.$$
$$(x-3)(x+4)=x^2+x-12.$$
$$(x+6)(x-9)=x^2-3x-54.$$
$$(x-4y)(x-10y)=x^2-14xy+40y^2.$$
$$(x-6y)(x+4y)=x^2-2xy-24y^2.$$

By an easy extension of these principles we may write down the product of *any* two binomials.

Thus
$$\begin{aligned}(2x+3y)(x-y)&=2x^2+3xy-2xy-3y^2\\&=2x^2+xy-3y^2.\\(3x-4y)(2x+y)&=6x^2-8xy+3xy-4y^2\\&=6x^2-5xy-4y^2.\\(x+4)(x-4)&=x^2+4x-4x-16\\&=x^2-16.\\(2x+5y)(2x-5y)&=4x^2+10xy-10xy-25y^2\\&=4x^2-25y^2.\end{aligned}$$

EXAMPLES V. f.

Write down the values of the following products:

1. $(a+3)(a-2)$.
2. $(a-7)(a-6)$.
3. $(x-4)(x+5)$.
4. $(b-6)(b+4)$.
5. $(y-7)(y-1)$.
6. $(a-1)(a-9)$.
7. $(c-5)(c+4)$.
8. $(x-9)(x-3)$.
9. $(y-4)(y+7)$.
10. $(a-3)(a+3)$.
11. $(x-5)(x-8)$.
12. $(a+7)(a-7)$.
13. $(k-6)(k-6)$.
14. $(a-5)(a+5)$.
15. $(c+7)(c+7)$.
16. $(p+9)(p-10)$.
17. $(z+5)(z-8)$.
18. $(x-9)(x+9)$.
19. $(x-3a)(x+2a)$.
20. $(a-2b)(a+2b)$.
21. $(x-4y)(x-4y)$.
22. $(a+4c)(a+4c)$.
23. $(c-5d)(c-5d)$.
24. $(p-2q)(p+2q)$.
25. $(2x-3)(3x+2)$.
26. $(3x-1)(2x+1)$.
27. $(5x-2)(5x+2)$.
28. $(3x+2a)(3x-2a)$.
29. $(6x+a)(6x-2a)$.
30. $(7x+3y)(7x-y)$.

CHAPTER VI.

DIVISION.

49. THE object of division is to find out the quantity, called the **quotient,** by which the **divisor** must be multiplied so as to produce the **dividend.**

Division is thus the inverse of multiplication.

The above statement may be briefly written

$$\text{quotient} \times \text{divisor} = \text{dividend},$$

or

$$\text{dividend} \div \text{divisor} = \text{quotient}.$$

It is sometimes better to express this last result as a fraction; thus

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient}.$$

Example 1. Since the product of 4 and x is $4x$, it follows that when $4x$ is divided by x the quotient is 4,

or otherwise, $4x \div x = 4$.

Example 2. Divide $27a^5$ by $9a^3$.

$$\text{The quotient} = \frac{27a^5}{9a^3} = \frac{27aaaaa}{9aaa}$$
$$= 3aa = 3a^2$$

We remove from the divisor and dividend the factors common to both, just as in arithmetic.

Therefore $27a^5 \div 9a^3 = 3a^2$.

Example 3. Divide $35a^3b^2c^3$ by $7ab^2c^2$.

$$\text{The quotient} = \frac{35aaa \,.\, bb \,.\, ccc}{7a \,.\, bb \,.\, cc} = 5aa \,.\, c = 5a^2c.$$

In each of these cases it should be noticed that the index of any letter in the quotient is the difference of the indices of that letter in the dividend and divisor.

50. It is easy to prove that *the rule of signs holds for division.*

Thus
$$ab \div a = \frac{ab}{a} = \frac{a \times b}{a} = b.$$
$$-ab \div a = \frac{-ab}{a} = \frac{a \times (-b)}{a} = -b.$$
$$ab \div (-a) = \frac{ab}{-a} = \frac{(-a) \times (-b)}{-a} = -b.$$
$$-ab \div (-a) = \frac{-ab}{-a} = \frac{(-a) \times b}{-a} = b.$$

Hence in division as well as multiplication

like signs produce $+$,
unlike signs produce $-$.

Rule. To divide one simple expression by another:

The index of each letter in the quotient is obtained by subtracting the index of that letter in the divisor from that in the dividend.

To the result so obtained prefix with its proper sign the quotient of the coefficient of the dividend by that of the divisor.

Example 1. Divide $84a^5x^3$ by $-12a^4x$.

The quotient $= (-7) \times a^{5-4}x^{3-1}$ $= -7ax^2$	Or at once mentally, $84a^5x^3 \div (-12a^4x) = -7ax^2$.

Example 2. $-45a^6b^2x^4 \div (-9a^3bx^2) = 5a^3bx^2$.

Note. If we apply the rule to divide any power of a letter by the same power of the letter we are led to a curious conclusion.

Thus, by the rule $\quad a^3 \div a^3 = a^{3-3} = a^0$;

but also $\quad a^3 \div a^3 = \dfrac{a^3}{a^3} = 1,$

$$\therefore \; a^0 = 1.$$

This result will appear somewhat strange to the beginner, but its full significance is explained in the Theory of Indices.

[See *Elementary Algebra*, Chap. XXXI.]

Rule. *To divide a compound expression by a single factor, divide each term separately by that factor, and take the algebraic sum of the partial quotients so obtained.*

This follows at once from Art. 38.

Examples. $(9x - 12y + 3z) \div (-3) = -3x + 4y - z.$

$(36a^3b^2 - 24a^2b^5 - 20a^4b^2) \div 4a^2b = 9ab - 6b^4 - 5a^2b.$

EXAMPLES VI. a.

Divide

1. $2x^3$ by x^2. 2. $6a^5$ by $3a$. 3. $5a^7$ by a^4.
4. $21b^7$ by $7b^3$. 5. x^3y^2 by $-xy$. 6. $-3xy^3$ by $3y$.
7. $4p^2q^3$ by $-2pq$. 8. $15m^3n$ by $-5m$. 9. $-l^3m^2$ by $-lm$.
10. $-48x^9$ by $-6x^3$. 11. $35z^{11}$ by $-7z^5$. 12. $-7a^2b$ by $-7b$.
13. $-28p^5q$ by $28p^5$. 14. $-7x^8$ by $-x^7$. 15. $24xyz^3$ by $-3z^2$.
16. $-12b^2c^5$ by $6b^2c^5$. 17. $-9k^{11}$ by $-k^{11}$. 18. $2k^3l^5$ by $-kl$.
19. $-45a^4b^3c^{15}$ by $9a^2b^3c^{10}$. 20. $-x^3y^4z^5$ by $-x^3yz^5$.
21. $-168a^2b^2cx^2$ by $-7abx^2$. 22. $-35a^6b^6x^7$ by $-7a^2b^4x^7$.
23. $3x^2-2x$ by x. 24. $5a^3b-7ab^3$ by ab.
25. $48p^2q-24pq^2$ by $8pq$. 26. $-15x^5+25x^4$ by $-5x^3$.
27. $x^2-xy-xz$ by $-x$. 28. $10a^3-5a^2b+a$ by $-a$.
29. $4x^3+36ax^2-16x$ by $-4x$. 30. $3a^3-9a^2b-6ab^2$ by $-3a$.

When the Divisor is a Compound Expression.

51. **Rule.** 1. *Arrange divisor and dividend in ascending or descending powers of some common letter.*

2. *Divide the term on the left of the dividend by the term on the left of the divisor, and put the result in the quotient.*

3. *Multiply the* WHOLE *divisor by this quotient, and put the product under the dividend.*

4. *Subtract and bring down from the dividend as many terms as may be necessary.*

Repeat these operations till all the terms from the dividend are brought down.

Example 1. Divide $x^2+11x+30$ by $x+6$.

Arrange the work thus:

$$x+6\;)\;x^2+11x+30\;($$

divide x^2, the first term of the dividend, by x, the first term of the divisor; the quotient is x. Multiply the *whole* divisor by x, and put the product x^2+6x under the dividend. We then have

$$\begin{array}{rl} x+6\;)\; & x^2+11x+30\;(\;x \\ & \underline{x^2+\;\,6x} \\ \text{by subtraction,} & 5x+30 \end{array}$$

On repeating the process above explained, we find that the next term in the quotient is $+5$.

The entire operation is more compactly written as follows:

$$\begin{array}{r|l} x+6\ )\ x^2+11x+30\ (\ x+5 & \\ x^2+\ 6x\qquad\ \ \ & \\ \hline 5x+30 & \\ 5x+30 & \\ \hline \end{array}$$

The reason for the rule is this: the dividend is separated into as many parts as may be convenient, and the complete quotient is found by taking the sum of all the partial quotients. By the above process $x^2+11x+30$ is separated into two parts, namely x^2+6x, and $5x+30$, and each of these is divided by $x+6$; thus we obtain the partial quotients $+x$ and $+5$.

Example 2. Divide $24x^2-65xy+21y^2$ by $8x-3y$.

$$\begin{array}{r} 8x-3y\ )\ 24x^2-65xy+21y^2\ (\ 3x-7y \\ 24x^2-\ 9xy\qquad\qquad\qquad \\ \hline -56xy+21y^2\qquad\quad \\ -56xy+21y^2\qquad\quad \\ \hline \end{array}$$

Divide $24x^2$ by $8x$; this gives $3x$, the first term of the quotient. Multiply the whole divisor by $3x$, and place the result under the dividend. By subtraction we obtain $-56xy+21y^2$. Divide the first term of this by $8x$, and so obtain $-7y$, the second term of the quotient.

Example 3. Divide $16a^3-46a^2+39a-9$ by $8a-3$.

$$\begin{array}{r} 8a-3\ )\ 16a^3-46a^2+39a-9\ (\ 2a^2-5a+3 \\ 16a^3-\ 6a^2\qquad\qquad\qquad\qquad\qquad \\ \hline -40a^2+39a\qquad\qquad\qquad \\ -40a^2+15a\qquad\qquad\qquad \\ \hline 24a-9\qquad\qquad\qquad \\ 24a-9\qquad\qquad\qquad \\ \hline \end{array}$$

Thus the quotient is $2a^2-5a+3$.

EXAMPLES VI. b.

Divide

1. a^2+2a+1 by $a+1$.
2. b^2+3b+2 by $b+2$.
3. x^2+4x+3 by $x+1$.
4. y^2+5y+6 by $y+3$.
5. x^2+5x-6 by $x-1$.
6. x^2+2x-8 by $x-2$.
7. $p^2+3p-40$ by $p+8$.
8. $q^2-4q-32$ by $q+4$.
9. $a^2+5a-50$ by $a+10$.
10. $m^2+7m-78$ by $m-6$.
11. $x^2+ax-30a^2$ by $x+6a$.
12. $a^2+9ab-36b^2$ by $a+12b$.

Divide

13. $-x^2+18x-45$ by $x-15$.
14. $x^2 - 42x+441$ by $x-21$.
15. $2x^2-13x-24$ by $2x+3$.
16. $5x^2+16x+3$ by $x+3$.
17. $6x^2+5x-21$ by $2x-3$.
18. $12a^2+ax-6x^2$ by $3a-2x$.
19. $-5x^2+xy+6y^2$ by $-x-y$.
20. $6a^2 - ac-35c^2$ by $2a-5c$.
21. $12p^2-74pq+12q^2$ by $2p-12q$.
22. $4m^2-49n^2$ by $2m+7n$.
23. $12a^2-31ab+20b^2$ by $4a-5b$.
24. $-25x^2+49y^2$ by $-5x+7y$.
25. $21p^2+11pq-40q^2$ by $3p+5q$.
26. $8x^3+8x^2+4x+1$ by $2x+1$.
27. $-2x^3+13x^2-17x+10$ by $-x+5$.
28. $x^3+ax^2-3a^2x-6a^3$ by $x-2a$.
29. $6x^3y-x^2y^2-7xy^3+12y^4$ by $2x+3y$.
30. $8x^3-12x^2-14x+21$ by $2x-3$.

52. The process of Art. 51 is applicable to cases in which the divisor consists of more than two terms.

Example 1. Divide $a^4-2a^3-7a^2+8a+12$ by a^2-a-6.

$$
\begin{array}{r}
a^2-a-6\;)\;a^4-2a^3-7a^2+8a+12\;(\;a^2-a-2 \\
\underline{a^4-a^3-6a^2} \\
-a^3-a^2+8a \\
\underline{-a^3+a^2+6a} \\
-2a^2+2a+12 \\
\underline{-2a^2+2a+12}
\end{array}
$$

Example 2. Divide $4x^3-5x^2+6x^5-15-x^4-x$ by $3+2x^2-x$.

First arrange each of the expressions in descending powers of x.

$$
\begin{array}{r}
2x^2-x+3\;)\;6x^5-x^4+4x^3-5x^2-x-15\;(\;3x^3+x^2-2x-5 \\
\underline{6x^5-3x^4+9x^3} \\
2x^4-5x^3-5x^2 \\
\underline{2x^4-x^3+3x^2} \\
-4x^3-8x^2-x \\
\underline{-4x^3+2x^2-6x} \\
-10x^2+5x-15 \\
-10x^2+5x-15
\end{array}
$$

Example 3. Divide $23x^2 - 2x^4 - 4x^3 + 12 + x^5 - 31x$ by $x^3 - 7x + 5$.

$$\begin{array}{r|l}
x^3 - 7x + 5 \,) & x^5 - 2x^4 - 4x^3 + 23x^2 - 31x + 12 \,(\, x^2 - 2x + 3 \\
& \underline{x^5 \qquad - 7x^3 + 5x^2} \\
& \quad - 2x^4 + 3x^3 + 18x^2 - 31x \\
& \quad \underline{- 2x^4 \qquad + 14x^2 - 10x} \\
& \qquad\qquad 3x^3 + 4x^2 - 21x + 12 \\
& \qquad\qquad \underline{3x^3 \qquad - 21x + 15} \\
& \qquad\qquad\qquad 4x^2 \qquad - 3
\end{array}$$

Now $4x^2$ is not divisible by x^3, so that the division cannot be carried on any further; thus the quotient is $x^2 - 2x + 3$, and there is a remainder $4x^2 - 3$.

In all cases where the division is not exact, the work should be carried on until the highest power in the remainder is lower than that in the divisor.

53. Occasionally it may be found convenient to arrange the expressions in *ascending* powers of some common letter.

Example. Divide $2a^3 + 10 - 16a - 39a^2 + 15a^4$ by $2 - 4a - 5a^2$.

$$\begin{array}{r|l}
2 - 4a - 5a^2 \,) & 10 - 16a - 39a^2 + 2a^3 + 15a^4 \,(\, 5 + 2a - 3a^2 \\
& \underline{10 - 20a - 25a^2} \\
& \qquad 4a - 14a^2 + 2a^3 \\
& \qquad \underline{4a - 8a^2 - 10a^3} \\
& \qquad\quad - 6a^2 + 12a^3 + 15a^4 \\
& \qquad\quad - 6a^2 + 12a^3 + 15a^4
\end{array}$$

EXAMPLES VI. c.

Divide

1. $a^3 - 6a^2 + 11a - 6$ by $a^2 - 4a + 3$.
2. $x^3 - 4x^2 + x + 6$ by $x^2 - x - 2$.
3. $y^3 + y^2 - 9y + 12$ by $y^2 - 3y + 3$.
4. $21m^3 - m^2 + m - 1$ by $7m^2 + 2m + 1$.
5. $6a^3 - 5a^2 - 9a - 2$ by $2a^2 - 3a - 1$.
6. $6k^3 - k^2 - 14k + 3$ by $3k^2 + 4k - 1$.
7. $6x^3 + 11x^2 - 39x - 65$ by $3x^2 + 13x + 13$.
8. $12x^3 - 8ax^2 - 27a^2x + 18a^3$ by $6x^2 - 13ax + 6a^2$.

Divide

9. $16x^3+14x^2y-129xy^2-15y^3$ by $8x^2+27xy+3y^2$.
10. $21c^3-5c^2d-3cd^2-2d^3$ by $7c^2+3cd+d^2$.
11. $3x^4-10x^3+12x^2-11x+6$ by $3x^2-x+3$.
12. $30a^4+11a^3-82a^2-12a+48$ by $3a^2+2a-4$.
13. $x^3-x^2-8x-13$ by x^2+3x+3.
14. $a+3a^3+6-10a^2$ by a^2-4a+3.
15. $21m^3-27m-26m^2+20$ by $3m+7m^2-4$.
16. $18x^3+24a^3-40a^2x-9ax^2$ by $9x^2+7a^2-18ax$.
17. $3y^4-4y^3+10y^2+3y-2$ by y^3-y^2+3y+2.
18. $5a^3+1+10a^4-4a^2$ by $5a^3-2a+1$.
19. $12x^4+5x^3-33x^2-3x+16$ by $4x^2-x-5$.
20. $p^4-6p^3+13p^2-10p+7$ by p^2-3p+2.
21. $28x^4+69x+2-71x^3-35x^2$ by $4x^2+6-13x$.
22. $5a^5-7a^4-9a^3-11a^2-38a+40$ by $-5a^2+17a-10$.
23. x^3-8a^3 by $x^2+2ax+4a^2$.
24. y^4+9y^2+81 by y^2-3y+9.
25. x^4+4y^4 by $x^2+2xy+2y^2$.
26. $9a^4-4a^2+4$ by $3a^2-4a+2$.
27. a^8+64 by a^4-4a^2+8.
28. $16x^4+36x^2+81$ by $4x^2+6x+9$.
29. $4m^5-29m-36+8m^2-7m^3+6m^4$ by m^3-2m^2+3m-4.
30. $15x^4+22-32x^3-30x+50x^2$ by $3-4x+5x^2$.
31. $3a^2+8ab+4b^2+10ac+8bc+3c^2$ by $a+2b+3c$.
32. $9x^2-4y^2+4yz-z^2$ by $-3x+2y-z$.
33. $4c^2-12c-d^2+9$ by $2c+d-3$.
34. $9p^2-16q^2+30p+25$ by $-3p-4q-5$.
35. $x^5-x^4y+x^3y^2-x^3+x^2-y^3$ by x^3-x-y.
36. $x^5+x^4y-x^3y^2+x^3-2xy^2$ by x^2+xy-y^2.
37. $a^3b^3+ab-9-b^4+3b^3+3b-a^4-3a^3-3a$ by $3-b+a^3$.
38. x^8+1 by x^3+x^2+x+1.
39. $2a^6+2$ by a^3+2a^2+2a+1.
40. $x^9-6x^4-8x^3-1$ by x^3-2x-1.

CHAPTER VII.

Removal and Insertion of Brackets.

54. Quantities are sometimes enclosed within brackets to indicate that they must all be operated upon in the same way. Thus in the expression $2a-3b-(4a-2b)$ the brackets indicate that the expression $4a-2b$ *treated as a whole* has to be subtracted from $2a-3b$.

It will be convenient here to quote the rules for removing brackets which have already been given in Arts. 24 and 25.

When an expression within brackets is preceded by the sign $+$, *the brackets can be removed without making any change in the expression.*

When an expression within brackets is preceded by the sign $-$, *the brackets may be removed if the sign of every term within the brackets be changed.*

Example. Simplify, by removing brackets, tne expression

$$(2a-3b)-(3a+4b)-(b-2a).$$

The expression $= 2a-3b-3a-4b-b+2a$

$= \quad a-8b$, by collecting like terms.

55. Sometimes it is convenient to enclose within brackets part of an expression already enclosed within brackets. For this purpose it is usual to employ brackets of different forms. The brackets in common use are ( ), { }, [ ].

56. When there are two or more pairs of brackets to be removed, it is generally best to begin with the innermost pair. In dealing with each pair in succession we apply the rules quoted above.

Example. Simplify, by removing brackets, the expression

$$a-2b-[4a-6b-\{3a-c+(2a-4b+c)\}].$$

Removing the brackets one by one,

$$\begin{aligned}\text{the expression} &= a-2b-[4a-6b-\{3a-c+2a-4b+c\}]\\ &= a-2b-[4a-6b-3a+c-2a+4b-c]\\ &= a-2b-4a+6b+3a-c+2a-4b+c\\ &= 2a, \quad \text{by collecting like terms.}\end{aligned}$$

Note. At first the beginner will find it best not to collect terms until all the brackets have been removed.

EXAMPLES VII. a.

Simplify by removing brackets and collecting like terms :

1. $a+2b+(2a-3b)$.
2. $a+2b-(2a-3b)$.
3. $2a-3b-(2a+2b)$.
4. $a-2-(4-3a)$.
5. $(x-3y)+(2x-4y)-(x-8y)$.
6. $a+2b-3c-(b-a-4c)$.
7. $(x-3y+2z)-(z-4y+2x)$.
8. $4x-(2y+2x)-(3x-5y)$.
9. $2a+(b-3a)-(4a-8b)-(6b-5a)$.
10. $m-(n-p)-(2m-2p+3n)-(n-m+2p)$.
11. $a-b+c-(a+c-b)-(a+b+c)-(b+c-a)$.
12. $5x-(7y+3x)-(2y+7x)-(3x+8y)$.
13. $(p-q)-(q-2p)+(2p-q)-(p-2q)$.
14. $2x^2-(3y^2-x^2)-(x^2-4y^2)$.
15. $(m^2-2n^2)-(2n^2-3m^2)-(3m^2-4n^2)$.
16. $(x-2a)-(x-2b)-\{2a-x-(2b+x)\}$.
17. $(a+3b)-(b-3a)-\{a+2b-(2a-b)\}$.
18. $p^2-2q^2-(q^2+2p^2)-\{p^2+3q^2-(2p^2-q^2)\}$.
19. $x-[y+\{x-(y-x)\}]$.
20. $(a-b)-\{a-b-(a+b)-(a-b)\}$.
21. $p-[p-(q+p)-\{p-(2p-q)\}]$.
22. $3x-y-[x-(2y-z)-\{2x-(y-z)\}]$.
23. $3a^2-[6a^2-\{8b^2-(9c^2-2a^2)\}]$.
24. $[3a-\{2a-(a-b)\}]-[4a-\{3a-(2a-b)\}]$.

57. A coefficient placed before any bracket indicates that every term of the expression within the bracket is to be multiplied by that coefficient; but when there are two or more brackets to be considered, a prefixed coefficient must be used as a multiplier only when its own bracket is being removed.

Examples 1. $2x+3(x-4)=2x+3x-12=5x-12$.

2. $7x-2(x-4)=7x-2x+8=5x+8$.

Example 3. Simplify $5a-4[10a+3\{x-a-2(a+x)\}]$.

The expression

$$\begin{aligned}&=5a-4[10a+3\{x-a-2a-2x\}]\\&=5a-4[10a+3\{-x-3a\}]\\&=5a-4[10a-3x-9a]\\&=5a-4[a-3x]\\&=5a-4a+12x\\&=a+12x.\end{aligned}$$

On removing the innermost bracket each term is multiplied by -2. Then before multiplying by 3, the expression within its bracket is simplified. The other steps will be easily seen.

58. Sometimes a line called a **vinculum** is drawn over the symbols to be connected; thus $a-\overline{b+c}$ is used with the same meaning as $a-(b+c)$, and hence $a-\overline{b+c}=a-b-c$.

NOTE. The line between the numerator and denominator of a fraction is a kind of vinculum. Thus $\frac{x-5}{3}$ is equivalent to $\frac{1}{3}(x-5)$.

Example 4. Find the value of

$$84-7[-11x-4\{-17x+3(8-\overline{9-5x})\}].$$

$$\begin{aligned}
\text{The expression } &= 84-7[-11x-4\{-17x+3(8-9+5x)\}]\\
&= 84-7[-11x-4\{-17x+3(5x-1)\}]\\
&= 84-7[-11x-4\{-17x+15x-3\}]\\
&= 84-7[-11x-4\{-2x-3\}]\\
&= 84-7[-11x+8x+12]\\
&= 84-7[-3x+12]\\
&= 84+21x-84\\
&= 21x.
\end{aligned}$$

When the beginner has had a little practice the number of steps may be considerably diminished.

Insertion of Brackets.

59. The rules for insertion of brackets are the converse of those given on page 12, and may be easily deduced from them.

For the following equivalents have been established in Arts. 24 and 25:

$$a+b-c=a+(b-c),$$
$$a-b-c=a-(b+c),$$
$$a-b+c=a-(b-c).$$

From these results the rules follow.

Rule. 1. *Any part of an expression may be enclosed within brackets and the sign + prefixed, the sign of every term within the brackets remaining unaltered.*

Examples. $a-b+c-d-e=a-b+(c-d-e).$
$$x^2-ax+bx-ab=(x^2-ax)+(bx-ab).$$

Rule. 2. *Any part of an expression may be enclosed within brackets and the sign − prefixed, provided the sign of every term within the brackets be changed.*

Examples. $a-b+c-d-e=a-(b-c)-(d+e).$
$$xy-ax-by+ab=(xy-by)-(ax-ab).$$

60. The terms of an expression can be bracketed in various ways.

Example. The expression $ax - bx + cx - ay + by - cy$

may be written $\qquad (ax - bx) + (cx - ay) + (by - cy),$

or $\qquad (ax - bx + cx) - (ay - by + cy),$

or $\qquad (ax - ay) - (bx - by) + (cx - cy).$

61. When every term of an expression is divisible by a common factor, the expression may be simplified by dividing each term by this factor, and enclosing the quotient within brackets, the common factor being placed outside as a coefficient.

Thus $\qquad 3x - 21 = 3(x - 7)\,;$

and $\qquad x^2 - 2ax + 4a^2 = x^2 - 2a(x - 2a).$

EXAMPLES VII. b.

Simplify by removing brackets:

1. $3(x - 2y) - 2(x - 4y).$ **2.** $x - 3(y - x) - 4(x - 2y).$

3. $16 - 3(2x - 3) - (2x + 3).$ **4.** $4(x + 3) - 2(7 + x) + 2.$

5. $8(x - 3) - (6 - 2x) - 2(x + 2) + 5(5 - x).$

6. $2x - 5(3x - 7 + y) + 4(2x + 3y - 8) - 7y.$

7. $2x - 5\{3x - 7(4x - 9)\}.$

8. $x^3 + 3(x^2y + xy^2) + y^3 - x^3 - 3(x^2y - xy^2) - y^3.$

9. $4x - 3\{x - (1 - y) + 2(1 - x)\}.$

10. $x - (y - z) - [x - y - z - 2(y + z)].$

11. $a^2 - [x^2 - \{x^2 - (z^2 - x^2 - y^2) - 2y^2\} + y^2].$

12. $5x + 4(y - 2z) - 4\{x + 2(y - z)\}.$

13. $a + \{-2b + 3(c - \overline{d - c})\}.$

14. $\{a^2 - (b^2 - c^2)\} - [2a^2 - \{a^2 - (b^2 - c^2)\} - 2(b^2 - c^2)].$

15. $3p - \{5q - [6q + 2(10q - p)]\}.$

16. $3x - 2[2x - \{2(x - y) - y\} - y].$

17. $3(5 - 6x) - 5[x - 5\{1 - 3(x - 5)\}].$

18. $12 - [6a - (7 - a - 5) - \{5a + (3 - \overline{2 - a})\}].$

19. $b^2 - \{a^2 + ab - (a^2 + b^2)\} - [a^2 - \{3ab - (b^2 - a^2)\}].$

20. $2[4x - \{2y + (2x - y) - (x + y)\}] - 2(-x - \overline{y - x}).$

21. $20(2 - x) + 3(x - 7) - 2[x + 9 - 3\{9 - 4(2 - x)\}].$

22. $-4(a + y) + 24(b - x) - 2[x + y + a - 3\{y + a - 4(b + x)\}].$

23. Multiply

$$2x-3y-4(x-2y)+5\{3x-2(x-y)\}$$
$$\text{by } 4x-(y-x)-3\{2y-3(x+y)\}.$$

In each of the following expressions bracket the powers of x so that the signs before the brackets may be (1) positive, (2) negative.

24. $ax^4+2x^3-cx^2+2x^2-bx^3-x^4.$

25. $ax^2+a^2x^3-bx^2-5x^2-cx^3.$

MISCELLANEOUS EXAMPLES II.

1. Find the sum of $a-2b+c$, $3b-(a-c)$, $3a-b+3c$.

2. Subtract $1-x^2$ from 1, and add the result to $2y-x^2$.

3. Simplify $a+2b-3c+(b-3a+2c)-(3b-2a-2c)$.

4. Find the continued product of $3x^2y$, $2xy^2$, $-7xy^3$, $-5x^4y^5$.

5. What quantity must be added to $p+q$ to make $2q$? And what must be added to p^2-3pq to make $p^2+2pq+q^2$?

6. Divide $1-6x^4+5x^3$ by $1-x+3x^2$.

---

7. Multiply $3b^2+2a^2-5ab$ by $2a+3b$.

8. When $x=2$, find the value of $1-x+x^2-\dfrac{x^3}{1+x}$.

9. Find the algebraic sum of $3ax$, $-2xz$, $9ax$, $-7xz$, $4ax$, $-4xz$.

10. Simplify $9a-(2b-c)+2d-(5a+3b)+4c-2d$, and find its value when $a=7$, $b=-3$, $c=-4$.

11. Subtract ax^2-4 from nothing, and add the difference to the sum of $2x^3-5x$ and unity.

12. Multiply $3x^2y-4xy^3z+2x^3y^2z^3$ by $-6x^2y^2z^3$ and divide the result by $3xy^2z^2$.

---

13. Simplify by removing brackets $5[x-4\{x-3(2x-\overline{3x+2})\}]$.

14. Simplify $2x^2-(2xy-3y^2)+4y^2+(5xy-2x^2)+x^2-(2xy+6y^2)$.

15. Find the product of $2x-7y$ and $3x+8y$, and multiply the result by $x+2y$.

16. Find the sum of $3a+2b$, $-5c-2d$, $3e+5f$, $b-a+2d$, $-2a-3b+5c-2f$.

17. Divide $x^4-4x^3-18x^2-11x+2$ by x^2-7x+1.

18. If $a=-1$, $b=2$, $c=0$, $d=1$, find the value of
$$ad+ac-a^2-cd+c^2-a+2c+a^2b+2a^3.$$

19. Simplify $3[1-2\{1-4(1-3x)\}]$, and find what quantity must be added to it to produce $3-8x$.

20. Divide the sum of $10x^2-7x(1+x^2)$ and $3(x^4+x^2+2)$ by $3(x^2+1)-(x+1)$.

21. Simplify $5x^4-8x^3-(2x^2-7)-(x^4+5)+(3x^3-x)$, and subtract the result from $4x^4-x+2$.

22. If $a=0$, $b=1$, $c=3$, $d=-2$, $e=2$, find the value of
(1) $3c^b-d^e$; (2) $(c+a)(c-a)+b^2$; (3) $e+a^b$.

23. Find the product of $7x^2-y(x-2y)$ and $x(7x+y)-2y^2$.

24. Subtract $(a^3+4)+(a^2-2)$ from $(a^3+4)(a^2-2)$.

---

25. Express by means of symbols
(1) b's excess over c is greater than a by 7.
(2) Three times the sum of a and $2b$ is less by 5 than the product of b and c.

26. Simplify
$$3a^2-(4a-b^2)-\{2a^2-(3b-a^2)-\overline{2b-3a}\}-\{5b-7a-(c^2-b^2)\}.$$

27. Find the continued product of
$$x^2+xy+y^2, \quad x^2-xy+y^2, \quad x^4-x^2y^2+y^4.$$

28. Divide $4a^2-9b^2-4ac+c^2$ by $2a-3b-c$.

29. If $a=3$, $b=-2$, $c=0$, $d=2$, find the value of
(1) $c(a+b)+b(a+c)+a(c-b)$; (2) a^a+d^d.

30. From a rod $a+b$ inches long $b-c$ inches are cut off; how much remains?

---

31. A boy buys a marbles, wins b, and loses c; how many has he then?

32. Simplify $2a-\{5a-[8a-(2b+a)]\}$, and find the value of $(a-b)[a^2+b(a+b)]$ when $a=1$, $b=2$.

33. Divide $1-5x^4+4x^5$ by x^2-2x+1.

34. Multiply the sum of $3x^2-5xy$ and $2xy-y^2$ by the excess of $3x^2+y^2$ over $2y^2+3xy$.

35. Express in algebraical symbols
(1) Three times x diminished by the sum of y and twice z.
(2) Seven times a taken from three times b is equal to five times the product of c and d.
(3) The sum of m and n multiplied by their difference is equal to the difference of the squares of m and n.

36. If $a=2$, $b=1$, $c=0$, $d=-1$, find the value of
$$(d-b)(c-b)+(ac-bd)^2+(c^2-d)(2c-b).$$

CHAPTER VIII.

Revision of Elementary Rules.

[If preferred, this chapter may be postponed until the chapters on Simple Equations and Problems have been read.]

Substitutions.

62. Definition. The **square root** of any proposed expression is that quantity whose square, or second power, is equal to the given expression. Thus the square root of 81 is 9, because $9^2 = 81$.

The square root of a is denoted by $\sqrt[2]{a}$, or more simply $\sqrt{a}$.

Similarly the **cube, fourth, fifth,** &c., **root** of any expression is that quantity whose third, fourth, fifth, &c., power is equal to the given expression.

The roots are denoted by the symbols $\sqrt[3]{}$, $\sqrt[4]{}$, $\sqrt[5]{}$, &c.

Examples. $\sqrt[3]{27} = 3$; because $3^3 = 27$.

$\sqrt[5]{32} = 2$; because $2^5 = 32$.

The root symbol $\sqrt{}$ is also called the **radical sign.**

Example 1. Find the value of $5\sqrt{(6a^3b^4c)}$, when $a = 3$, $b = 1$, $c = 8$.

$$\begin{aligned} 5\sqrt{(6a^3b^4c)} &= 5 \times \sqrt{(6 \times 3^3 \times 1^4 \times 8)} \\ &= 5 \times \sqrt{(6 \times 27 \times 8)} \\ &= 5 \times \sqrt{(3 \times 27) \times (2 \times 8)} \\ &= 5 \times 9 \times 4 \\ &= 180. \end{aligned}$$

Note. An expression of the form $\sqrt{(6a^3b^4c)}$ is often written $\sqrt{6a^3b^4c}$, the line above being used as a vinculum indicating the square root of the expression *taken as a whole.*

Example 2. If $a = -4$, $b = -3$, $c = -1$, $f = 0$, $x = 4$, find the value of

$$7\sqrt[3]{(a^2cx)} - 3\sqrt{b^4c^2} + 5\sqrt{(f^2x)}.$$

$$\begin{aligned} \text{The expression} &= 7\sqrt[3]{(-4)^2(-1)4} - 3\sqrt{(-3)^4(-1)^2} + 0 \\ &= 7\sqrt[3]{(-64)} - 3\sqrt{81} \\ &= 7 \times (-4) - 3 \times 9 \\ &= -55. \end{aligned}$$

EXAMPLES VIII. a.

If $a=4,\ b=1,\ c=6,\ d=0$, find the value of

1. $\sqrt{b^4}$. 2. $\sqrt{9ab}$. 3. $\sqrt{6b^3c}$. 4. $\sqrt{9a^3b^2}$.
5. $\sqrt{4b^4c^2}$. 6. $\sqrt{6a^4b^2c}$. 7. $a\sqrt[3]{9ac}$. 8. $3b\sqrt[3]{3a^2c^2}$.
9. $\sqrt{a^3b^3}-\sqrt{9c^2}$. 10. $3\sqrt{a^3cd^2}-d\sqrt[3]{2a^2b}+\sqrt{6ac}$.

If $a=-3,\ b=2,\ c=-1,\ x=-4,\ y=0$, find the value of

11. $\sqrt{a^2cx}$. 12. $\sqrt{3ac^3}$. 13. $\sqrt{6abx}$. 14. $5\sqrt{c^3x}$.
15. $\sqrt{3ab^2c^2x}$. 16. $\sqrt[3]{a^3c^2}$. 17. $\sqrt[3]{bc^3x}$. 18. $\sqrt[3]{3a^2b^3c}$.
19. $\sqrt{3ac}-\sqrt{cx}+\sqrt{b^2cx}$. 20. $\sqrt{c^2y}+\sqrt{2a^2b}-\sqrt{9a^2}$.
21. If $x=100,\ y=81,\ z=16$, find the value of

$$\sqrt{\frac{x}{4}}-\sqrt[4]{y}+\sqrt[3]{4z}.$$

22. If $a=-6,\ b=2,\ c=-1,\ x=-4,\ y=0$, find the value of

$$2\sqrt{a^2cx}-2\sqrt{a^2b^4x^2y^6}+\sqrt{8a^2b}.$$

Fractional Coefficients and Indices.

63. Fractional Coefficients. The rules which have been already explained in the case of integral coefficients are still applicable when the coefficients are fractional.

Example 1. Find the sum of $\frac{2}{3}x^2+\frac{1}{3}xy-\frac{1}{4}y^2$, $-x^2-\frac{2}{3}xy+2y^2$, $\frac{2}{3}x^2-xy-\frac{5}{4}y^2$.

$$\begin{array}{r}
\frac{2}{3}x^2+\frac{1}{3}xy-\frac{1}{4}y^2\\
-\ \ x^2-\frac{2}{3}xy+2y^2\\
\frac{2}{3}x^2-\ \ xy-\frac{5}{4}y^2\\
\hline
\frac{1}{3}x^2-\frac{4}{3}xy+\frac{1}{2}y^2
\end{array}$$

Here each column is added up separately, and the fractional coefficients combined by the rules of arithmetic.

Example 2. Divide $\frac{1}{4}x^3+\frac{1}{72}xy^2+\frac{1}{12}y^3$ by $\frac{1}{2}x+\frac{1}{3}y$.

$$\begin{array}{l}
\frac{1}{2}x+\frac{1}{3}y\ )\ \frac{1}{4}x^3+\frac{1}{72}xy^2+\frac{1}{12}y^3\ (\ \frac{1}{2}x^2-\frac{1}{3}xy+\frac{1}{4}y^2\\
\qquad\qquad\ \ \frac{1}{4}x^3+\frac{1}{6}x^2y\\
\hline
\qquad\qquad\qquad -\frac{1}{6}x^2y+\frac{1}{72}xy^2\\
\qquad\qquad\qquad -\frac{1}{6}x^2y-\frac{1}{9}xy^2\\
\hline
\qquad\qquad\qquad\qquad\qquad \frac{1}{8}xy^2+\frac{1}{12}y^3\\
\qquad\qquad\qquad\qquad\qquad \frac{1}{8}xy^2+\frac{1}{12}y^3
\end{array}$$

64. Fractional Indices. In all the examples hitherto explained the indices have been integers, but expressions involving fractional and negative indices such as $a^{\frac{2}{3}}$, $x^{-\frac{1}{2}}$, $3x^{\frac{3}{4}}+x^{\frac{1}{4}}-2$, $a^{-2}-4a^{-1}x-3x^2$ may be dealt with by the same rules. For a complete discussion of the theory of Indices the student is referred to the *Elementary Algebra*, Chap. XXXI. It will be sufficient here to point out that the rules for combination of indices in multiplication and division given in Chapters V. and VI. are universally true.

Example 1. $x^{\frac{2}{3}}\times x^{\frac{3}{4}}=x^{\frac{2}{3}+\frac{3}{4}}=x^{\frac{17}{12}}$.

Example 2. $a^{-4}\times a^{4}=a^{-4+4}=a^{0}=1$. [See Note, Art. 50.]

Example 3. $2a^{\frac{1}{2}}b^{-1}\times 3a^{-\frac{1}{2}}b^{\frac{1}{4}}=6a^{\frac{1}{2}-\frac{1}{2}}b^{-1+\frac{1}{4}}=6a^{0}b^{-\frac{3}{4}}=6b^{-\frac{3}{4}}$.

Example 4. $3x^{2}y^{\frac{3}{4}}\div x^{3}y^{\frac{1}{2}}=3x^{2-3}y^{\frac{3}{4}-\frac{1}{2}}=3x^{-1}y^{\frac{1}{4}}$.

Example 5. $a^{-2}b^{\frac{1}{3}}\div a^{2}b^{-\frac{1}{2}}=a^{-2-2}b^{\frac{1}{3}+\frac{1}{2}}=a^{-4}b^{\frac{5}{6}}$.

It will be seen from these illustrations that the rules for combining indices in multiplication and division may be concisely expressed by the two statements,

(1) $a^m\times a^n=a^{m+n}$, (2) $a^m\div a^n=a^{m-n}$;

where m and n may have any values positive or negative, integral or fractional.

65. We shall now give some examples involving compound expressions.

Example 1. Multiply $x^{\frac{2}{3}}-3x^{\frac{1}{3}}+4$ by $2x^{\frac{1}{3}}-1$.

$$\begin{array}{rrrr}
 & x^{\frac{2}{3}} & -3x^{\frac{1}{3}} & +4 \\
 & & 2x^{\frac{1}{3}} & -1 \\
\hline
2x & -6x^{\frac{2}{3}} & +8x^{\frac{1}{3}} & \\
 & -x^{\frac{2}{3}} & +3x^{\frac{1}{3}} & -4 \\
\hline
2x & -7x^{\frac{2}{3}} & +11x^{\frac{1}{3}} & -4
\end{array}$$

Example 2. Multiply $c^x+2c^{-x}-7$ by $5-3c^{-x}+2c^x$.

$$\begin{array}{rrrrr}
c^x & -7 & +2c^{-x} & & \\
2c^x & +5 & -3c^{-x} & & \\
\hline
2c^{2x} & -14c^x & +4 & & \\
 & +5c^x & -35 & +10c^{-x} & \\
 & & -3 & +21c^{-x} & -6c^{-2x} \\
\hline
2c^{2x} & -9c^x & -34 & +31c^{-x} & -6c^{-2x}
\end{array}$$

Here the expressions have been arranged in descending powers of c, and it should be noticed that in this arrangement the numerical terms -7 and $+5$ stand between the terms involving c^x and c^{-x}.

Example 3. Divide

$$24x^{\frac{1}{4}} - 16x^{-\frac{3}{4}} + x^{\frac{7}{4}} - 16x^{-\frac{1}{4}} - 5x^{\frac{5}{4}} \text{ by } 8x^{-\frac{1}{4}} - 2x^{\frac{3}{4}} + x^{\frac{5}{4}} - 4x^{\frac{1}{4}}.$$

Arrange divisor and dividend in descending powers of x.

$$\begin{array}{r|l}
x^{\frac{5}{4}} - 2x^{\frac{3}{4}} - 4x^{\frac{1}{4}} + 8x^{-\frac{1}{4}} & x^{\frac{7}{4}} - 5x^{\frac{5}{4}} + 24x^{\frac{1}{4}} - 16x^{-\frac{1}{4}} - 16x^{-\frac{3}{4}} \;\big(\, x^{\frac{1}{2}} - 3 - 2x^{-\frac{1}{2}} \\
 & \underline{x^{\frac{7}{4}} - 2x^{\frac{5}{4}} - 4x^{\frac{3}{4}} + 8x^{\frac{1}{4}}} \\
 & -3x^{\frac{5}{4}} + 4x^{\frac{3}{4}} + 16x^{\frac{1}{4}} - 16x^{-\frac{1}{4}} \\
 & \underline{-3x^{\frac{5}{4}} + 6x^{\frac{3}{4}} + 12x^{\frac{1}{4}} - 24x^{-\frac{1}{4}}} \\
 & -2x^{\frac{3}{4}} + 4x^{\frac{1}{4}} + 8x^{-\frac{1}{4}} - 16x^{-\frac{3}{4}} \\
 & \underline{-2x^{\frac{3}{4}} + 4x^{\frac{1}{4}} + 8x^{-\frac{1}{4}} - 16x^{-\frac{3}{4}}}
\end{array}$$

EXAMPLES VIII. b.

1. Find the sum of $-\frac{1}{3}m - \frac{1}{4}n$, $-\frac{2}{3}m + \frac{3}{4}n$, $-2m - n$.
2. Add together $\frac{2}{3}a - \frac{1}{6}b + \frac{1}{4}c$, $\frac{1}{2}a - \frac{1}{2}b$, $\frac{1}{6}a + \frac{1}{12}b + \frac{1}{4}c$, $-\frac{1}{3}a + \frac{7}{12}b - \frac{1}{2}c$.
3. From $a + \frac{1}{2}b - \frac{1}{3}c$ take $\frac{1}{3}a - b + \frac{1}{2}c$.
4. Subtract $\frac{1}{4}a^2 + \frac{1}{3}ab - \frac{1}{4}b^2$ from $\frac{1}{2}a^2 - \frac{1}{3}ab + \frac{1}{4}b^2$.
5. Multiply $\frac{1}{3}x^2 + \frac{1}{2}y^2$ by $\frac{1}{2}x - \frac{1}{3}y$.
6. Find the product of $\frac{1}{2}x^2 - \frac{1}{3}x + \frac{1}{4}$ and $\frac{1}{2}x + \frac{1}{3}$.
7. Divide $\frac{2}{9}x^3 - \frac{3}{4}y^3$ by $\frac{1}{3}x - \frac{1}{2}y$.
8. Divide $a^3 - 2a^2b + \frac{11}{9}ab^2 - \frac{2}{9}b^3$ by $a^2 - \frac{5}{3}ab + \frac{2}{3}b^2$.
9. Simplify $\frac{1}{4}(2x - 3y) - \frac{1}{3}(3x + 2y) + \frac{1}{12}(7x - 5y)$.
10. Find the sum of

$$\tfrac{1}{4}y^3 - \tfrac{1}{12}y^2 + \tfrac{3}{7}y - \tfrac{1}{3},\ \ \tfrac{1}{2}y^2 + \tfrac{1}{6} + \tfrac{1}{14}y,\ \ -\tfrac{1}{6}y^2 - \tfrac{1}{4}y + \tfrac{5}{12}.$$

11. Find the product of

$$\tfrac{1}{2}x - \tfrac{1}{3}y + \tfrac{1}{3}(z - \tfrac{1}{2}y) \text{ and } \tfrac{1}{3}(x - z) - \tfrac{1}{2}(y - \tfrac{1}{3}x).$$

12. Simplify by removing brackets

$$8\left(\frac{a}{4} - \frac{b}{2}\right) + 5\left\{2a - 3\left(a - \frac{b}{3}\right)\right\}.$$

13. Divide $\frac{1}{3}x^3 - \frac{17}{36}x^2 + \frac{1}{3}x - \frac{1}{8}$ by $\frac{2}{3}x - \frac{1}{2}$.
14. Subtract $\frac{1}{12}(7x - 9y)$ from $\frac{1}{3}(x - 3y) - \frac{1}{2}(y - 2x)$.
15. Add together $(x - \frac{1}{2}y)(\frac{1}{3}x + y)$ and $(2x - \frac{1}{3}y)(\frac{1}{2}x - y)$.
16. Multiply $\frac{2}{3}a^3 - \frac{4}{5}a^2x + \frac{1}{2}x^3$ by $\frac{3}{5}a - 2x$.
17. Divide $36a^2 + \frac{1}{9}b^2 + \frac{1}{4} - 4ab - 6a + \frac{1}{3}b$ by $6a - \frac{1}{3}b - \frac{1}{2}$.
18. Simplify $6\{x - \frac{3}{2}(y - \frac{4}{3})\}\{\frac{1}{2}(2x - y) + 2(y - 1)\}$.

19. Multiply $\frac{3}{2}a^2-\frac{1}{2}ab+b^2$ by $a^2+\frac{1}{3}ab-\frac{2}{3}b^2$, and verify the result when $a=1$, $b=2$.

20. Multiply $x-x^{\frac{1}{2}}y^{\frac{1}{2}}+y$ by $x^{\frac{1}{2}}-y^{\frac{1}{2}}$.

21. Divide $x^{\frac{4}{3}}+x^{\frac{2}{3}}y^{\frac{1}{2}}+y$ by $x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$.

22. Find the product of $x^{\frac{1}{4}}y+y^{\frac{2}{3}}$ and $x^{\frac{1}{3}}-y^{\frac{1}{3}}$.

23. Multiply $a^{\frac{3}{2}}-x^{\frac{3}{2}}$ by $a^{\frac{2}{3}}+x^{\frac{2}{3}}$.

24. Divide $c^{-3}-8c^{-1}-3$ by $c^{-1}-3$.

25. Divide

$$4x^{\frac{2}{3}}y^{-2}-12x^{\frac{1}{3}}y^{-1}+25-24x^{-\frac{1}{3}}y+16x^{-\frac{2}{3}}y^2 \text{ by } 2x^{\frac{1}{3}}y^{-1}-3+4x^{-\frac{1}{3}}y.$$

26. Find the value of $(ax^{-2}+a^{-1}x)(ax^{-2}-3a^{-1}x)$.

27. Find the square of $a^{\frac{1}{4}}-1-a^{-\frac{1}{4}}$.

28. Find the continued product of $3a^{-2}b^{-1}x$, $ax^{\frac{1}{3}}-b^{\frac{2}{3}}$, and $ax^{\frac{2}{3}}+b$.

29. Divide $x-y$ by $x^{\frac{5}{6}}y^{\frac{1}{4}}+x^{\frac{1}{2}}y^{\frac{7}{12}}+x^{\frac{1}{6}}y^{\frac{11}{12}}$.

30. Multiply $a^2+2a^{-2}-7$ by $5+a^2-2a^{-2}$.

31. Find the value of $(3x^ay^{-a}-x^{-a}y^a)(x^ay-x^{-a}y^{-1})$.

Important Cases in Division.

66. The following example in division is worthy of notice.

Example. Divide $a^3+b^3+c^3-3abc$ by $a+b+c$.

$$\begin{array}{l}
a+b+c\,)\,a^3-3abc+b^3+c^3\,(\,a^2-ab-ac+b^2-bc+c^2\\
\qquad\qquad\underline{a^3+a^2b+a^2c}\\
\qquad\qquad\quad -a^2b-a^2c-3abc\\
\qquad\qquad\quad \underline{-a^2b-ab^2-abc}\\
\qquad\qquad\qquad\quad -a^2c+ab^2-2abc\\
\qquad\qquad\qquad\quad \underline{-a^2c-abc-ac^2}\\
\qquad\qquad\qquad\qquad\quad ab^2-abc+ac^2+b^3\\
\qquad\qquad\qquad\qquad\quad \underline{ab^2+b^3+b^2c}\\
\qquad\qquad\qquad\qquad\qquad -abc+ac^2-b^2c\\
\qquad\qquad\qquad\qquad\qquad \underline{-abc-b^2c-bc^2}\\
\qquad\qquad\qquad\qquad\qquad\qquad ac^2+bc^2+c^3\\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{ac^2+bc^2+c^3}
\end{array}$$

Here the work is arranged in descending powers of a, and the other letters are taken alphabetically; thus in the first remainder a^2b precedes a^2c, and a^2c precedes $3abc$. A similar arrangement will be observed throughout the work.

67. The following examples in division may be easily verified; they are of great importance and should be carefully noticed.

$$\text{I.}\quad \begin{cases} \dfrac{x^2-y^2}{x-y}=x+y, \\[2mm] \dfrac{x^3-y^3}{x-y}=x^2+xy+y^2, \\[2mm] \dfrac{x^4-y^4}{x-y}=x^3+x^2y+xy^2+y^3, \end{cases}$$

and so on; the divisor being $x-y$, the terms in the quotient *all positive*, and the index in the dividend *either odd or even*.

$$\text{II.}\quad \begin{cases} \dfrac{x^3+y^3}{x+y}=x^2-xy+y^2, \\[2mm] \dfrac{x^5+y^5}{x+y}=x^4-x^3y+x^2y^2-xy^3+y^4, \\[2mm] \dfrac{x^7+y^7}{x+y}=x^6-x^5y+x^4y^2-x^3y^3+x^2y^4-xy^5+y^6, \end{cases}$$

and so on; the divisor being $x+y$, the terms in the quotient *alternately positive and negative*, and the index in the dividend *always odd*.

$$\text{III.}\quad \begin{cases} \dfrac{x^2-y^2}{x+y}=x-y, \\[2mm] \dfrac{x^4-y^4}{x+y}=x^3-x^2y+xy^2-y^3, \\[2mm] \dfrac{x^6-y^6}{x+y}=x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5, \end{cases}$$

and so on; the divisor being $x+y$, the terms in the quotient *alternately positive and negative*, and the index in the dividend *always even*.

IV. The expressions x^2+y^2, x^4+y^4, x^6+y^6, ... (where the index is *even*, and the terms *both positive*) are *never* exactly divisible by $x+y$ or by $x-y$.

All these different cases may be more concisely stated as follows:

(1) x^n-y^n is divisible by $x-y$ if n be *any* whole number.
(2) x^n+y^n is divisible by $x+y$ if n be any *odd* whole number.
(3) x^n-y^n is divisible by $x+y$ if n be any *even* whole number.
(4) x^n+y^n is never divisible by $x+y$ or by $x-y$, when n is an *even* whole number.

Dimension and Degree.

68. Each of the letters composing a term is called a **dimension** of the term, and the number of letters involved is called the **degree** of the term. Thus the product abc is said to be *of three dimensions*, or *of the third degree*; and ax^4 is said to be *of five dimensions*, or *of the fifth degree*.

A numerical coefficient is not counted. Thus $8a^2b^5$ and a^2b^5 are each of *seven* dimensions.

69. The **degree of an expression** is the degree of the term of highest dimensions contained in it; thus a^4-8a^3+3a-5 is *an expression of the fourth degree*, and $a^2x-7b^2x^3$ is *an expression of the fifth degree*. But it is sometimes useful to speak of the dimensions of an expression with regard to some one of the letters it involves. For instance the expression ax^3-bx^2+cx-d is said to be of *three dimensions in x*.

70. A compound expression is said to be **homogeneous** when all its terms are of the same degree. Thus $8a^6-a^4b^2+9ab^5$ is a *homogeneous expression of the sixth degree*.

It is useful to notice that the product of two homogeneous expressions is also homogeneous.

Thus by Art. 47,

$$(2a^2-3ab+4b^2)(-5a^2+3ab+4b^2)=-10a^4+21a^3b-21a^2b^2+16b^4.$$

Here the product of two homogeneous expressions each of two dimensions is a homogeneous expression of four dimensions.

Also the quotient of one homogeneous expression by another homogeneous expression is itself homogeneous.

For instance in the example of Art. 66 it may be noticed that the divisor is homogeneous of one dimension, the dividend is homogeneous of three dimensions, and the quotient is homogeneous of two dimensions.

EXAMPLES VIII. c.

1. Divide $a^3+30ab-125b^3+8$ by $a-5b+2$.
2. Divide $x^3+y^3-z^3+3xyz$ by $x+y-z$.
3. Divide $a^3-b^3+1+3ab$ by $a-b+1$.
4. Divide $18cd+1+27c^3-8d^3$ by $1+3c-2d$.

Without actual division write down the quotients in the following cases :

5. $\dfrac{x^3-1}{x-1}$.   6. $\dfrac{a^3+b^3}{a+b}$.   7. $\dfrac{x^4-a^4}{x-a}$.   8. $\dfrac{x^4-a^4}{x+a}$

9. $\dfrac{1+a^3}{1+a}$.   10. $\dfrac{16-b^4}{2+b}$.   11. $\dfrac{a^5+b^5}{a+b}$.   12. $\dfrac{a^5-b^5}{a-b}$

13. $\dfrac{x^3+27y^3}{x+3y}$.   14. $\dfrac{a^6-x^6}{a+x}$.   15. $\dfrac{c^7+1}{c+1}$.   16. $\dfrac{x^6+y^6}{x^2+y^2}$

17. In the expression

$$2a^3b^2+3ab^4+3a^2b^2x-x^5+20a^2b^3-11a^4+7a^3b^2,$$

which terms are *like*, and which are *homogeneous*?

18. In each term of the expression

$$7a^3bc^2-ab^2c+12b^3c^4-b^5c,$$

introduce some power of a which will make the whole expression homogeneous of the eighth degree.

19. By considering the dimensions of the product, correct the following statement

$$(3x^2-5xy+y^2)(8x^2-2xy-3y^2)=24x^4-46x^2y+9x^2y^2+13xy^3-3y^3,$$

it being known that there is no mistake in the *coefficients*.

20. Write down the square of $3a^2-2ab-b^2$, having given that the coefficients of the terms taken in descending powers of a are 9, -12, -2, 4, 1.

21. Write down the value of the product of $3a^2b+5a^3-ab^2$ and $ab^2+5a^3-3a^2b$, having given that the coefficients of the terms when arranged in ascending powers of b are 25, 0, -9, 6, -1.

22. The quotient of $x^3-y^3-1-3xy$ by $x-y-1$ is

$$x^2+xy+x+y^2-y+1.$$

Introduce the letter z into dividend, divisor, and quotient so as to make them respectively homogeneous expressions of three, one, and two dimensions.

CHAPTER IX.

SIMPLE EQUATIONS.

71. AN **equation** asserts that two expressions are equal, but we do not usually employ the word equation in so wide a sense.

Thus the statement $x+3+x+4=2x+7$, which is *always* true whatever value x may have, is called an **identical equation,** or briefly an **identity.**

The parts of an equation to the right and left of the sign of equality are called **members** or **sides** of the equation, and are distinguished as the *right side* and *left side*.

72. Certain equations are only true for particular values of the symbols employed. Thus $3x=6$ is only true when $x=2$, and is called an **equation of condition,** or more usually an equation. Consequently an *identity* is an equation which is *always* true whatever be the values of the symbols involved; whereas an **equation** (in the ordinary use of the word) is only true for *particular* values of the symbols. In the above example $3x=6$, the value 2 is said to **satisfy** the equation. The object of the present chapter is to explain how to treat an equation of the simplest kind in order to discover the value which satisfies it.

73. The letter whose value it is required to find is called the **unknown quantity.** The process of finding its value is called **solving the equation.** The value so found is called the **root** or the **solution** of the equation.

74. An equation which involves the unknown quantity in the first degree is called a **simple equation.** It is usual to denote the unknown quantity by the letter x.

The process of solving a simple equation depends only upon the following **axioms:**

1. If to equals we add equals the sums are equal.
2. If from equals we take equals the remainders are equal.
3. If equals are multiplied by equals the products are equal.
4. If equals are divided by equals the quotients are equal.

75. Consider the equation $7x=14$.

It is required to find what numerical value x must have to *satisfy* this equation.

Dividing both sides by 7 we get

$$x=2, \qquad \text{[Axiom 4]}.$$

Similarly, if $\frac{x}{2}=-6$,

multiplying both sides by 2, we get

$$x=-12, \qquad \text{[Axiom 3]}.$$

Again, in the equation $7x-2x-x=23+15-10$, by collecting terms, we have $4x=28$.

$$\therefore\ x=7.$$

Transposition of Terms.

76. To solve $3x-8=x+12$.

This case differs from the preceding in that the unknown quantity occurs on both sides of the equation. We can, however, **transpose** any term from one side to the other by simply *changing its sign*. This we proceed to show.

Subtract x from both sides of the equation, and we get

$$3x-x-8=12, \qquad \text{[Axiom 2]}.$$

Adding 8 to both sides, we have

$$3x-x=12+8, \qquad \text{[Axiom 1]}.$$

Thus we see that $+x$ has been removed from one side, and appears as $-x$ on the other; and -8 has been removed from one side and appears as $+8$ on the other.

Hence we may enunciate the following rule:

Rule. *Any term may be transposed from one side of the equation to the other by changing its sign.*

It appears from this that *we may change the sign of every term in an equation;* for this is equivalent to transposing all the terms, and then making the right and left hand members change places.

Example. Take the equation $-3x-12=x-24$.

Transposing, $-x+24=3x+12$,

or $3x+12=-x+24$,

which is the original equation with the sign of every term changed.

77. To solve $\frac{x}{2}-3=\frac{x}{4}+\frac{x}{5}$.

Here it will be convenient to begin by clearing the equation of *fractional* coefficients. This can always be done by multiplying both sides of the equation by the least common multiple of the denominators. [Axiom 3.]

Thus, multiplying by 20,

$$10x-60=5x+4x\ ;$$

transposing, $$10x-5x-4x=60\ ;$$

$$\therefore\ x=60.$$

78. We can now give a general rule for solving any simple equation with one unknown quantity.

Rule. *First, if necessary, clear of fractions; then transpose all the terms containing the unknown quantity to one side of the equation, and the known quantities to the other. Collect the terms on each side; divide both sides by the coefficient of the unknown quantity, and the value required is obtained.*

Example 1. Solve $5(x-3)-7(6-x)+3=24-3(8-x)$.

Removing brackets, $5x-15-42+7x+3=24-24+3x\ ;$

transposing, $$5x+7x-3x=24-24+15+42-3\ ;$$

$$\therefore\ 9x=54\ ;$$

$$\therefore\ x=6.$$

Example 2. Solve $(x+1)(2x-1)-5x=(2x-3)(x-5)+47$.

Forming the products, we have

$$2x^2+x-1-5x=2x^2-13x+15+47.$$

Erasing the term $2x^2$ on each side, and transposing,

$$x-5x+13x=15+47+1\ ;$$

$$\therefore\ 9x=63\ ;$$

$$\therefore\ x=7.$$

79. It is extremely useful for the beginner to acquire the habit of **verifying**, that is, proving the truth of his results; the habit of applying such tests tends to make the student self-reliant and confident in his own accuracy.

In the case of simple equations we have only to show that when we substitute the value of x in the two sides of the equation we obtain the same result.

Example. To show that $x=7$ *satisfies* the equation

$$(x+1)(2x-1)-5x=(2x-3)(x-5)+47.$$

When $x=7$, the left side $(x+1)(2x-1)-5x$

$$=(7+1)(14-1)-35=(8\times 13)-35=69.$$

The right side $(2x-3)(x-5)+47$

$$=(14-3)(7-5)+47=(11\times 2)+47=69.$$

Thus, since these two results are the same, $x=7$ satisfies the equation.

EXAMPLES IX. a.

Write down the solutions of the following equations:

1. $7x=21$.
2. $3x=15$.
3. $9x=18$.
4. $5x=5$.
5. $12x=132$.
6. $33=11x$.
7. $4x=-12$.
8. $-10=-5x$.
9. $4x=18$.
10. $12x=42$.
11. $30=-6x$.
12. $4x=0$.
13. $6x=26$.
14. $0=11x$.
15. $1=11x$.
16. $3x=-27$.
17. $0=-2x$.
18. $6x=3$.
19. $5=15x$.
20. $-24=-8x$.

Solve the following equations:

21. $6x+3=15$.
22. $5x-7=28$.
23. $13=7+2x$.
24. $15=37-11x$.
25. $4x-7=11$.
26. $7x=18-2x$.
27. $3x-18=7-2x$.
28. $4x=13-2x-10$.
29. $3x=7-2x+8$.
30. $0=11-2x+7-10x$.
31. $8x-3-5x-5=7x$.
32. $7x-13=12-5x-5$.
33. $5x-17+3x-5=6x-7-8x+115$.
34. $7x-21-4x+13+2x=41-5x-7+6x$.
35. $15-7x-9x-28+14x-17=21-3x+13-9x+8x$.
36. $5x-6x+30-7x=2x+10-7x+5x-20$.
37. $5(x-3)=4(x-2)$.
38. $11(5-4x)=7(5-6x)$.
39. $3-7(x-1)=5-4x$.
40. $5-4(x-3)=x-2(x-1)$.
41. $8(x-3)-2(3-x)=2(x+2)-5(5-x)$.
42. $4(5-x)-2(x-3)=x-4-3(x+2)$.
43. $\frac{1}{2}x+\frac{1}{3}x=x-3$.
44. $\frac{1}{2}x-\frac{1}{3}x=\frac{1}{4}x+\frac{1}{2}$.
45. $x-\frac{x}{4}-\frac{1}{2}=3+\frac{x}{4}$.
46. $\frac{1}{2}x-\frac{3}{4}x-1\frac{1}{3}=\frac{1}{6}x+2$.
47. $(x+3)(2x-3)-6x=(x-4)(2x+4)+12$.
48. $(x+2)(x+3)+(x-3)(x-2)-2x(x+1)=0$.
49. $(2x+1)(2x+6)-7(x-2)=4(x+1)(x-1)-9x$.
50. $(3x+1)^2+6+18(x+1)^2=9x(3x-2)+65$.

51. Show that $x=5$ satisfies the equation
$$5x-6(x-4)=2(x+5)+5(x-4)-6.$$
52. Show that $x=15$ is the solution of the equation
$$7(25-x)-2x-15=2(3x-25)-x.$$
53. Verify that $x=3$ satisfies the equation
$$2(x+1)(x+3)+8=(2x+1)(x+5).$$
54. Show that $x=4$ satisfies the equation
$$(3x+1)(2x-7)=6(x-3)^2+7.$$

80. We shall now give some equations of greater difficulty.

Example 1. Solve $5x-(4x-7)(3x-5)=6-3(4x-9)(x-1)$.

Simplifying, we have
$$5x-(12x^2-41x+35)=6-3(4x^2-13x+9);$$
and by removing brackets
$$5x-12x^2+41x-35=6-12x^2+39x-27.$$
Erase the term $-12x^2$ on each side and transpose;

thus $$5x+41x-39x=6-27+35;$$
$$\therefore \quad 7x=14;$$
$$\therefore \quad x=2.$$

Note. Since the $-$ sign before a bracket affects every term within it, in the first line of work we do not remove the brackets until we have formed the products.

Example 2. Solve $4-\dfrac{x-9}{8}=\dfrac{x}{22}-\dfrac{1}{2}$.

Multiply by 88, the least common multiple of the denominators;
$$352-11(x-9)=4x-44;$$
removing brackets, $352-11x+99=4x-44$;

transposing, $-11x-4x=-44-352-99$;

collecting terms and changing signs, $15x=495$;
$$\therefore \quad x=33.$$

Note. In this equation $-\dfrac{x-9}{8}$ is regarded as a single term with the minus sign before it. In fact it is equivalent to $-\dfrac{1}{8}(x-9)$, the *vinculum* or line between the numerator and denominator having the same effect as a bracket. [Art. 58.]

In certain cases it will be found more convenient not to multiply throughout by the L.C.M. of the denominator, but to clear of fractions in two or more steps.

Example 3. Solve $\frac{x-4}{3}+\frac{2x-3}{35}=\frac{5x-32}{9}-\frac{x+9}{28}$.

Multiplying throughout by 9, we have

$$3x-12+\frac{18x-27}{35}=5x-32-\frac{9x+81}{28};$$

transposing, $$\frac{18x-27}{35}+\frac{9x+81}{28}=2x-20.$$

Now clear of fractions by multiplying by $5\times 7\times 4$ or 140;

$$\therefore\quad 72x-108+45x+405=280x-2800;$$

$$\therefore\quad 2800-108+405=280x-72x-45x;$$

$$\therefore\quad 3097=163x;$$

$$\therefore\quad x=19.$$

81. To solve equations whose coefficients are decimals, we may express the decimals as common fractions, and proceed as before; but it is often found more simple to work entirely in decimals.

Example. Solve $\cdot375x-1\cdot875=\cdot12x+1\cdot185$.

Transposing, $\cdot375x-\cdot12x=1\cdot185+1\cdot875;$

collecting terms, $(\cdot375-\cdot12)x=3\cdot06;$

that is, $\cdot255x=3\cdot06;$

$$\therefore\quad x=\frac{3\cdot06}{\cdot255}$$

$$=12.$$

EXAMPLES IX. b.

Solve the equations:

1. $(x+15)(x-3)-(x-3)^2=30-15(x-1)$.
2. $15-3x=(2x+1)(2x-1)-(2x-1)(2x+3)$.
3. $21-x(2x+1)+2(x-4)(x+2)=0$.
4. $3(x+5)-3(2x-1)=32-4(x-5)^2+4x^2$.
5. $3x^2-7x-(x+2)(x-2)=(x+1)(x-1)+(x-3)(x+3)$.
6. $(x-6)(2x-9)-(11-2x)(7-x)=5x-4-7(x-2)$.
7. $\frac{x-1}{5}+\frac{x-9}{2}=3$.
8. $\frac{x}{6}+\frac{x-8}{4}=1+\frac{x-6}{3}$.
9. $\frac{x+8}{3}=2+\frac{x-6}{7}$.
10. $\frac{6x-2}{9}+\frac{3x+5}{18}=\frac{1}{3}$.

Solve the equations:

11. $\frac{10x+1}{5}-1=5x-2.$ **12.** $x+3+\frac{x-2}{5}=7+2x.$

13. $\frac{x-6}{4}-\frac{x-4}{6}=1-\frac{x}{10}.$ **14.** $\frac{x+12}{6}-x=6\frac{1}{2}-\frac{x}{12}.$

15. $\frac{x+5}{6}-\frac{x+1}{9}=\frac{x+3}{4}.$ **16.** $\frac{11-6x}{5}-\frac{9-7x}{2}=\frac{5(x-1)}{6}.$

17. $\frac{47-6x}{5}-(x-6)=\frac{4(x-7)}{15}.$ **18.** $\frac{4-5x}{6}-\frac{1-2x}{3}=\frac{13}{42}.$

19. $\frac{3x-1}{10}-\frac{x-1}{4}=\frac{2x-31}{3}.$ **20.** $\frac{1-2x}{7}-\frac{2-3x}{8}=1\frac{1}{2}+\frac{x}{4}.$

21. $\frac{3}{4}(x-1)-\frac{5}{3}(x-4)=\frac{8}{5}(x-6)+\frac{5}{12}.$

22. $\frac{3}{5}(x-4)-\frac{1}{3}(2x-9)=\frac{1}{4}(x-1)-2.$

23. $\frac{1}{6}(x+4)-\frac{1}{2}(x-3)=\frac{1}{2}(3x-5)-\frac{1}{4}(x-6)-\frac{1}{5}(x-2).$

24. $\frac{1}{7}(3-8x)-\frac{1}{5}(7-2x)+\frac{x-1}{5}=2-x-\frac{1}{5}(1-6x).$

25. $\frac{1}{3}(x+4)-\frac{1}{9}(20-x)=\frac{1}{18}(5x-1)-\frac{1}{6}(5x-13)+8.$

26. $\frac{x+1}{2}-\frac{5x+9}{28}=\frac{x+6}{21}+5-\frac{x-12}{3}.$

27. $5-\frac{10x+1}{27}-\frac{x}{8}=\frac{13x+4}{18}-\frac{5(x-4)}{4}.$

28. $\frac{3x}{4}-\frac{x-7}{51}-(x-3)-\frac{6}{17}(x+10)+\frac{2x+5}{4}=0.$

29. $\frac{x+4}{39}-\frac{1}{5}(1-x)=2-\frac{3}{26}(6-5x)-\frac{1}{5}(x+4).$

30. $\frac{3}{11}+\frac{1}{44}x=\frac{1}{2}\left(\frac{4}{11}-\frac{x}{33}\right)-\frac{5}{66}+\frac{1}{3}\left(1-\frac{x}{22}\right).$

31. $\cdot7x-3\cdot35=6\cdot4-3\cdot2x.$ **32.** $\cdot5x+\cdot25+\cdot1+1\cdot25=\cdot4x.$

33. $3\cdot25x-\cdot75x=9+1\cdot5x.$ **34.** $\cdot2x-\cdot01x+\cdot005x=11\cdot7.$

35. $\cdot5x-\cdot\dot{6}x=\cdot75x-11.$ **36.** $\cdot4x-\cdot8\dot{3}x=\cdot7-\cdot\dot{3}.$

37. Find the value of x which makes the two expressions $(3x-1)(4x-11)$ and $6(2x-1)(x-3)$ equal.

38. What value of x will make the expression $77x-3(2x-1)(4x-2)$ equal to $337-8(3x-1)(x+1)$?

CHAPTER X.

SYMBOLICAL EXPRESSION.

82. IN solving algebraical problems the chief difficulty of the beginner is to express the conditions of the question by means of symbols. A question proposed in algebraical symbols will frequently be found puzzling, when a similar arithmetical question would present no difficulty. Thus, the answer to the question "find a number greater than x by a" may not be self-evident to the beginner, who would of course readily answer an analogous arithmetical question, "find a number greater than 50 by 6." The process of addition which gives the answer in the second case supplies the necessary hint; and, just as the number which is greater than 50 by 6 is $50+6$, so the number which is greater than x by a is $x+a$.

83. The following examples will perhaps be the best introduction to the subject of this chapter. After the first we leave to the student the choice of arithmetical instances, should he find them necessary.

Example 1. By how much does x exceed 17?

Take a numerical instance; "by how much does 27 exceed 17?"
The answer obviously is 10, which is equal to $27-17$.
Hence the excess of x over 17 is $x-17$.
Similarly the defect of x from 17 is $17-x$.

Example 2. If x is one *part* of 45 the other part is $45-x$.

Example 3. If x is one *factor* of 45 the other factor is $\frac{45}{x}$.

Example 4. How far can a man walk in a hours at the rate of 4 miles an hour?

In 1 hour he walks 4 miles.
In a hours he walks a times as far, that is, $4a$ miles.

Example 5. If \$20 is divided equally among y persons, the share of each is the total sum divided by the number of persons, or $\$\frac{20}{y}$.

Example 6. Out of a purse containing $\$x$ and y half-dollars a man spends z quarters; express in cents the sum left.

$$\$x = 4x \text{ quarters},$$

and $$y \text{ half-dollars} = 2y \text{ quarters};$$

$$\therefore \text{ the sum left} = (4x+2y-z) \text{ quarters},$$
$$= 25(4x+2y-z) \text{ cents}.$$

EXAMPLES X. a.

1. By how much does x exceed 5?
2. By how much is y less than 15?
3. What must be added to a to make 7?
4. What must be added to 6 to make b?
5. By what must 5 be multiplied to make a?
6. What is the quotient when 3 is divided by a?
7. By what must $6x$ be divided to get 2?
8. By how much does $6x$ exceed $2x$?
9. The sum of two numbers is x and one of the numbers is 10; what is the other?
10. The sum of three numbers is 100; if one of them is 25 and another is x, what is the third?
11. The product of two factors is $4x$; if one of the factors is 4, what is the other?
12. The product of two numbers is p, and one of them is m; what is the other?
13. How many times is x contained in $2y$?
14. The difference of two numbers is 8, and the greater of them is a; what is the other?
15. The difference of two numbers is x, and the less of them is 6; what is the other?
16. What number is less than 30 by y?
17. The sum of 12 equal numbers is $48x$; what is the value of each number?
18. How many numbers each equal to y must be taken to make $15xy$?
19. If there are x numbers each equal to $2a$, what is their sum?
20. If there are 5 numbers each equal to x, what is their product?

21. If there are x numbers each equal to p, what is their product?

22. If there are n books each worth y dollars, what is the total cost?

23. If n books of equal value cost x dollars, what does each cost?

24. How many books each worth two dollars can be bought for y dollars?

25. If apples are sold at x for a dime, what will be the cost in cents of y apples?

26. What is the price in cents of n oranges at six cents a score?

27. If I spend n dimes out of a sum of \$5, how many dimes have I left?

28. What is the daily wage in dimes of a man who earns \$12 in p weeks, working 6 days a week?

29. How many days must a man work in order to earn \$6 at the rate of y dimes a day?

30. If x persons combine to pay a bill of \$$y$, what is the share of each in dimes?

31. How many dimes must a man pay out of a sum of \$$p$ so as to have left $30x$ cents?

32. How many persons must contribute equally to a fund consisting of \$$x$, so that the subscription of each may equal y quarters?

33. How many hours will it take to travel x miles at 10 miles an hour?

34. How far can I walk in p hours at the rate of q miles an hour?

35. If I can walk m miles in n days, what is my rate per day?

36. How many days will it take to travel y miles at x miles a day?

84. We subjoin a few harder examples worked out in full.

Example 1. What is (1) the sum, (2) the product of three consecutive numbers of which the least is n?

The two numbers consecutive to n are $n+1$ and $n+2$;

$$\therefore \text{ the sum} = n + (n+1) + (n+2)$$
$$= 3n + 3.$$

And the product $= n(n+1)(n+2)$.

Example 2. A boy is x years old, and five years hence his age will be half that of his father: how old is the father now?

In five years the boy will be $x+5$ years old; therefore his father will then be $2(x+5)$, or $2x+10$ years old; his present age must therefore be $2x+10-5$ or $2x+5$ years.

Example 3. A and B are playing for money; A begins with \$$p$ and B with q dimes. B wins \$$x$; express by an equation the fact that A has now 3 times as much as B.

What B has won A has lost;

$\therefore$ A has $p-x$ dollars, that is $10(p-x)$ dimes,

B has q dimes $+x$ dollars, that is $q+10x$ dimes.

Thus the required equation is $10(p-x)=3(q+10x)$.

Example 4. A man travels a miles by coach and b miles by train; if the coach goes at the rate of 7 miles an hour, and the train at the rate of 25 miles per hour, how long does the journey take?

The coach travels 7 miles in 1 hour;

$\therefore$ .... 1 .... $\frac{1}{7}$ hour;

that is, ... a .... $\frac{a}{7}$ hours.

Similarly the train travels b miles in $\frac{b}{25}$ hours.

$\therefore$ the whole time occupied is $\frac{a}{7}+\frac{b}{25}$ hours.

Example 5. How many men will be required to do in p hours what q men do in np hours?

np hours is the time occupied by q men;

$\therefore$ 1 hour .... $q\times np$ men;

that is, p hours...... .... $\frac{q\times np}{p}$ men.

Therefore the required number of men is qn.

EXAMPLES X. b.

1. Write down three consecutive numbers of which a is the least.

2. Write down four consecutive numbers of which b is the greatest.

3. Write down five consecutive numbers of which c is the middle one.

4. What is the next odd number after $2n-1$?

5. What is the even number next before $2n$?

6. Write down the product of three odd numbers of which the middle one is $2x+1$.

7. How old is a man who will be x years old in 15 years?

8. How old was a man x years ago if his present age is n years?

9. In $2x$ years a man will be y years old, what is his present age?

10. How old is a man who in x years will be twice as old as his son now aged 20 years?

11. In 5 years a boy will be x years old; what is the present age of his father if he is twice as old as his son?

12. A has $\$m$ and B has n dimes; after A has won 3 dimes from B, each has the same amount. Express this in algebraical symbols.

13. A has 25 dollars and B has 13 dollars; after B has won x dollars he then has four times as much as A. Express this in algebraical symbols.

14. How many miles can a man walk in 30 minutes if he walks 1 mile in x minutes?

15. How many miles can a man walk in 50 minutes if he walks x miles in y minutes?

16. How long will it take a man to walk p miles if he walks 15 miles in q hours?

17. How far can a pigeon fly in x hours at the rate of 2 miles in 7 minutes?

18. A man travels x miles by boat and y miles by train, how long will the journey take if the train goes 30 miles and the boat 10 miles an hour?

19. If x men do a work in $5x$ hours, how many men will be required to do the same work in y hours?

20. How long will it take p men to mow q acres of corn, if each man mows r acres a day?

21. Write down a number which, when divided by a, gives a quotient b and remainder c.

22. What is the remainder if x divided by y gives a quotient z?

23. What is the quotient if when m is divided by n there is a remainder r?

24. If a bill is shared equally among n persons, and each pays 75 cents, how many dollars does the bill amount to?

25. A man has $\$x$ in his purse, he pays away 25 dimes, and receives y cents; express in dimes the sum he has left.

26. How many dollars does a man save in a year, if he earns $\$x$ a week and spends y quarters a calendar month?

27. What is the total cost of $6x$ nuts and $4x$ plums, when x plums cost a dime and plums are three times as expensive as nuts?

28. If on an average there are x words in a line, and y lines in a page, how many pages will be required for a book which contains z words?

CHAPTER XI.

PROBLEMS LEADING TO SIMPLE EQUATIONS.

85. THE principles of the last chapter may now be employed to solve various problems.

The method of procedure is as follows:

Represent the unknown quantity by a symbol, as x, and express in symbolical language the conditions of the question; we thus obtain a simple equation which can be solved by the methods already given in Chapter IX.

Example I. Find two numbers whose sum is 28, and whose difference is 4.

Let x be the smaller number, then $x+4$ is the greater.

Their sum is $x+(x+4)$, which is to be equal to 28.

Hence
$$x+x+4=28;$$
$$2x=24;$$
$$\therefore\quad x=12,$$
and
$$x+4=16,$$
so that the numbers are 12 and 16.

The beginner is advised to test his solution by finding whether it satisfies the conditions of the question or not.

Example II. Divide $47 between A, B, C, so that A may have $10 more than B, and B $8 more than C.

Let x represent the *number* of dollars that C has; then B has $x+8$ dollars, and A has $x+8+10$ dollars.

Hence
$$x+(x+8)+(x+8+10)=47;$$
$$x+x+8+x+8+10=47,$$
$$3x=21;$$
$$\therefore\quad x=7;$$
so that C has $7, B $15, A $25.

EXAMPLES XI. a.

1. Six times a number increased by 11 is equal to 65; find it.

2. Find a number which when multiplied by 11 and then diminished by 18 is equal to 15.

3. If 3 be added to a number, and the sum multiplied by 12, the result is 84; find the number.

4. One number exceeds another by 3, and their sum is 27; find them.

5. Find two numbers whose sum is 30, and such that one of them is greater than the other by 8.

6. Find two numbers which differ by 10, so that one is three times the other.

7. Find two numbers whose sum is 19, such that one shall exceed twice the other by 1.

8. Find two numbers whose sum shall be 26 and their difference 8.

9. Divide \$100 between *A* and *B* so that *B* may have \$30 more than *A*.

10. Divide \$66 between *A*, *B*, and *C* so that *B* may have \$8 more than *A*, and *C* \$14 more than *B*.

11. *A*, *B*, and *C* have \$72 among them; *C* has twice as much as *B*, and *B* has \$4 less than *A*; find the share of each.

12. How must a sum of 73 dollars be divided among *A*, *B*, and *C*, so that *B* may have 8 dollars less than *A* and 4 dollars more than *C*?

Example III. Divide 60 into two parts, so that three times the greater may exceed 100 by as much as 8 times the less falls short of 200.

Let x be the greater part, then $60-x$ is the less.

Three times the greater part is $3x$, and its excess over 100 is

$$3x-100.$$

Eight times the less is $8(60-x)$, and its defect from 200 is

$$200-8(60-x).$$

Whence the symbolical statement of the question is

$$3x-100=200-8(60-x);$$

$$3x-100=200-480+8x,$$

$$480-100-200=8x-3x,$$

$$5x=180;$$

$$\therefore \quad x=36, \text{ the greater part,}$$

and $$60-x=24, \text{ the less.}$$

Example V. A is 4 years older than B, and half A's age exceeds one-sixth of B's age by 8 years; find their ages.

Let x be the *number* of years in B's age, then A's age is $x+4$ years.

One-half of A's age is represented by $\frac{1}{2}(x+4)$ years, and one-sixth of B's age by $\frac{1}{6}x$ years.

Hence $$\tfrac{1}{2}(x+4)-\tfrac{1}{6}x=8;$$
multiplying by 6 $$3x+12-x=48;$$
$$\therefore\ 2x=36;$$
$$\therefore\ x=18.$$

Thus B's age is 18 years, and A's age is 22 years.

13. Divide 75 into two parts, so that three times one part may be double of the other.

14. Divide 122 into two parts, such that one may be as much above 72 as twice the other is below 60.

15. A certain number is doubled and then increased by 5, and the result is less by 1 than three times the number; find it.

16. How much must be added to 28 so that the resulting number may be 8 times the added part?

17. Find the number whose double exceeds its half by 9.

18. What is the number whose seventh part exceeds its eighth part by 1?

19. Divide 48 into two parts, so that one part may be three-fifths of the other.

20. If A, B, and C have \$76 between them, and A's money is double of B's and C's one-sixth of B's, what is the share of each?

21. Divide \$511 between A, B, and C, so that B's share shall be one-third of A's, and C's share three-fourths of A's and B's together.

22. B is 16 years younger than A, and one-half B's age is equal to one-third of A's; how old are they?

23. A is 8 years younger than B, and 24 years older than C; one-sixth of A's age, one-half of B's, and one-third of C's together amount to 38 years; find their ages.

24. Find two consecutive numbers whose product exceeds the square of the smaller by 7. [See Art. 81, Ex. 1.]

25. The difference between the squares of two consecutive numbers is 31; find the numbers.

86. We shall now give examples of somewhat greater difficulty.

Example I. A has \$6, and B has six dimes; after B has won from A a certain sum, A has then five-sixths of what B has; how much did B win?

Suppose that B wins x dimes, A has then $60-x$ dimes, and B has $6+x$ dimes.

Hence
$$60-x=\tfrac{5}{6}(6+x);$$
$$360-6x=30+5x,$$
$$11x=330;$$
$$x=30.$$

Therefore B wins 30 dimes, or \$3.

Example II. A is twice as old as B, ten years ago he was four times as old; what are their present ages?

Let B's age be x years, then A's age is $2x$ years.

Ten years ago their ages were respectively $x-10$ and $2x-10$ years; thus we have
$$2x-10=4(x-10);$$
$$2x-10=4x-40,$$
$$2x=30;$$
$$\therefore\ x=15,$$

so that B is 15 years old, A 30 years.

EXAMPLES XI. b.

1. A has \$12 and B has \$8; after B has lost a certain sum to A his money is only three-sevenths of A's; how much did A win?

2. A and B begin to play each with \$15; if they play till B's money is four-elevenths of A's, what does B lose?

3. A and B have \$28 between them; A gives \$3 to B and then finds he has six times as much money as B; how much had each at first?

4. A had three times as much money as B; after giving \$3 to B he had only twice as much; what had each at first?

5. A father is four times as old as his son; in 16 years he will only be twice as old; find their ages.

6. A is 20 years older than B, and 5 years ago A was twice as old as B; find their ages.

7. How old is a man whose age 10 years ago was three-eighths of what it will be in 15 years?

8. A is twice as old as B; 5 years ago he was three times as old; what are their present ages?

9. A father is 24 years older than his son; in 7 years the son's age will be two-fifths of his father's age; what are their present ages?

Example III. A person spent \$56.40 in buying geese and ducks; if each goose cost 7 dimes, and each duck 3 dimes, and if the total number of birds bought was 108, how many of each did he buy?

In questions of this kind it is of essential importance to have all quantities expressed in the same denomination; in the present instance it will be convenient to express the money in dimes.

Let x be the number of geese, then $108-x$ is the number of ducks.

Since each goose costs 7 dimes, x geese cost $7x$ dimes.

And since each duck costs 3 dimes, $108-x$ ducks cost $3(108-x)$ dimes.

Therefore the amount spent is

$$7x+3(108-x) \text{ dimes.}$$

But the question states that the amount is also \$56.40, that is 564 dimes.

Hence
$$7x+3(108-x)=564;$$
$$7x+324-3x=564,$$
$$4x=240,$$
$$\therefore\ x=60, \text{ the number of geese,}$$
and
$$108-x=48, \text{ the number of ducks.}$$

Note. In all these examples it should be noticed that the unknown quantity x represents a *number* of dollars, ducks, years, etc.; and the student must be careful to avoid beginning a solution with a supposition of the kind, "let $x=A$'s share" or "let $x=$the ducks," or any statement so vague and inexact.

It will sometimes be found easier not to put x equal to the quantity directly required, but to some other quantity involved in the question; by this means the equation is often simplified.

Example IV. A woman spends \$1 in buying eggs, and finds that 9 of them cost as much over 25 cents as 16 cost under 75 cents; how many eggs did she buy?

Let x be the price of an egg in cents; then 9 eggs cost $9x$ cents, and 16 eggs cost $16x$ cents;

$$\therefore\ 9x-25=75-16x,$$
$$25x=100;$$
$$\therefore\ x=4.$$

Thus the price of an egg is 4 cents, and the number of eggs $=100 \div 4=25$.

10. A sum of \$30 is divided between 50 men and women, the men each receiving 75 cents, and the women 50 cents; find the number of each sex.

11. The price of 13 yards of cloth is as much less than \$10 as the price of 27 yards exceeds \$20; find the price per yard.

12. A hundredweight of tea, worth \$68, is made up of two sorts, part worth 80 cents a pound and the rest worth 50 cents a pound; how much is there of each sort?

13. A man is hired for 60 days on condition that for each day he works he shall receive \$2, but for each day that he is idle he shall pay \$1 for his board: at the end he received \$90; how many days had he worked?

14. A sum of \$6 is made up of 30 coins, which are either quarters or dimes; how many are there of each?

15. A sum of \$11.45 was paid in half-dollars, quarters, and dimes; the number of half-dollars used was four times the number of quarters and ten times the number of dimes; how many were there of each?

16. A person buys coffee and tea at 40 cents and 80 cents a pound respectively; he spends \$15.10, and in all gets 24 lbs.; how much of each did he buy?

17. A man sold a horse for a sum of money which was greater by \$68 than half the price he paid for it, and gained thereby \$18; what did he pay for the horse?

18. Two boys have 240 marbles between them; one arranges his in heaps of 6 each, the other in heaps of 9 each. There are 36 heaps altogether; how many marbles has each?

19. A man's age is four times the combined ages of his two sons, one of whom is three times as old as the other; in 24 years their combined ages will be 12 years less than their father's age; find their respective ages.

20. A sum of money is divided between three persons, A, B, and C, in such a way that A and B have \$42 between them, B and C have \$45, and C and A have \$53; what is the share of each?

21. A person bought a number of oranges for \$3, and finds that 12 of them cost as much over 24 cents as 16 of them cost under 60 cents; how many oranges were bought?

22. By buying eggs at 15 for a quarter and selling them at a dozen for 15 cents a man lost \$1.50; find the number of eggs.

23. I bought a certain number of apples at four for a cent, and three-fifths of that number at three for a cent; by selling them at sixteen for five cents I gained 4 cents; how many apples did I buy?

24. If 8 lbs. of tea and 24 lbs. of sugar cost \$7.20, and if 3 lbs. of tea cost as much as 45 lbs. of sugar, find the price of each per pound.

25. Four dozen of port and three dozen of sherry cost \$89; if a bottle of port costs 25 cents more than a bottle of sherry, find the price of each per dozen.

26. A man sells 50 acres more than the fourth part of his farm and has remaining 10 acres less than the third; find the number of acres in the farm.

27. Find a number such that if we divide it by 10 and then divide 10 by the number and add the quotients, we obtain a result which is equal to the quotient of the number increased by 20 when divided by 10.

28. A sum of money is divided between three persons, A, B, and C, in such a way that A receives \$10 more than one-half of the entire amount, B receives \$10 more than one-third, and C the remainder, which is \$10; find the amounts received by A and B.

29. The difference between two numbers is 15, and the quotient arising from dividing the greater by the less is 4; find the numbers.

30. A person in buying silk found that if he should pay \$3.50 per yard he would lack \$15 of having money enough to pay for it; he therefore purchased an inferior quality at \$2.50 per yard and had \$25 left; how many yards did he buy?

31. Find two numbers which are to each other as 2 to 3, and whose sum is 100.

32. A man's age is twice the combined ages of his three sons, the eldest of whom is 3 times as old as the youngest and $\frac{3}{2}$ times as old as the second son; in 10 years their combined ages will be 4 years less than their father's age; find their respective ages.

33. The sum of \$34.50 was given to some men, women, and children, each man receiving \$2, each woman \$1, and each child 50 cents. The number of men was 4 less than twice the number of women, and the number of children was 1 more than twice the number of women; find the total number of persons.

34. A man bought a number of apples at the rate of 5 for 3 cents. He sold four-fifths of them at 4 for 3 cents and the remainder at 2 for a cent, gaining 10 cents; how many did he buy?

35. A farm of 350 acres was owned by four men, A, B, C, and D. B owns five-sixths as much as A, C four-fifths as much as B, and D one-sixth as much as A, B, and C together; find the number of acres owned by each.

CHAPTER XII.

ELEMENTARY FRACTIONS.

Highest Common Factor of Simple Expressions.

88. DEFINITION. The **highest common factor** of two or more algebraical expressions is the expression of highest dimensions [Art. 68] which divides each of them without remainder.

The abbreviation H.C.F. is sometimes used instead of the words *highest common factor*.

89. In the case of *simple expressions* the highest common factor can be written down by inspection.

Example 1. The highest common factor of a^4, a^3, a^2, a^6 is a^2.

Example 2. The highest common factor of a^3b^4, $a^2b^5c^2$, a^4b^7c is a^2b^4; for a^2 is the highest power of a that will divide a^3, a^2, a^4; b^4 is the highest power of b that will divide b^4, b^5, b^7; and c is not a *common* factor.

90. If the expressions have numerical coefficients, find by Arithmetic their greatest common measure, and prefix it as a coefficient to the algebraical highest common factor.

Example. The highest common factor of $21a^4x^3y$, $35a^2x^4y$, $28a^3xy$ is $7a^2xy$; for it consists of the product of

(1) the greatest common measure of the numerical coefficients;

(2) the highest power of each letter which divides every one of the given expressions.

EXAMPLES XII. a.

Find the highest common factor of

1. $3ab^2$, $2ab^3$. **2.** x^3y^2, $4x^2y^5$. **3.** bc^5, $5b^3c$. **4.** $4x^5$, $2xy^2z^3$.

5. a^2b^2c, a^3bc^5. **6.** $3a^2b$, $9abc$. **7.** $6x^2y^2z$, $2xy$. **8.** $15y^3$, $5xy^6z^2$

9. $12a^3bc^2$, $18ab^2c^3$. **10.** $7x^3y^5z^4$, $21x^2yz^3$.

11. $8ax$, $6a^2y$, $10ab^2x^2$. **12.** a^2x^3y, b^3xy^4, cx^4y^2.

13. $14bc^2$, $63ba^2$, $56b^2c$. **14.** $15x^2y$, $60x^5y^2z^3$, $25x^3z^4$.

15. $17xy^2z$, $51xyz^2$, $34x^2yz$. **16.** $77a^3b^5c^2$, $33a^2b^3c^5$, ab^2c^6.

Lowest Common Multiple of Simple Expressions.

91. Definition. The **lowest common multiple** of two or more algebraical expressions is the expression of lowest dimensions which is divisible by each of them without remainder.

The abbreviation L.C.M. is sometimes used instead of the words *lowest common multiple.*

92. In the case of *simple expressions* the lowest common multiple can be written down by inspection.

Example 1. The lowest common multiple of a^4, a^3, a^2, a^6 is a^6.

Example 2. The lowest common multiple of a^3b^4, ab^5, a^2b^7 is a^3b^7; for a^3 is the lowest power of a that is divisible by each of the quantities a^3, a, a^2; and b^7 is the lowest power of b that is divisible by each of the quantities b^4, b^5, b^7.

93. If the expressions have numerical coefficients, find by Arithmetic their least common multiple, and prefix it as a coefficient to the algebraical lowest common multiple.

Example. The lowest common multiple of $21a^4x^3y$, $35a^2x^4y$, $28a^3xy$ is $420a^4x^4y$; for it consists of the product of

(1) the least common multiple of the numerical coefficients;

(2) the lowest power of each letter which is divisible by every power of that letter occurring in the given expressions.

EXAMPLES XII. b.

Find the lowest common multiple of

1. xyz, $3y^3$. **2.** a^2b^4, abc. **3.** $2x^3y$, $3xy^2z$.
4. $4a^2$, $3abx^4$. **5.** $4a^4bc^3$, $5ab^2$. **6.** $2ab$, $4xy$.
7. mn, nl, lm. **8.** xy^2, $3yz^2$, $2zx^2$. **9.** $2xy$, $3yz$, $4zx$.
10. p^2qr, q^3p^2r, $7pq$. **11.** $15x^2y$, $25xyz^3$. **12.** $9ab^3$, $21a^2c$.
13. $27a^3$, $81b^3$, $18a^2b^5$. **14.** $5ax^6$, $6cy$, $7a^2x^3c^5z$.
15. $15a^2b^3$, $20ax^2y$, $30x^2$. **16.** $72p^2q^3r^4$, $108p^3q^2r$.

Find both the highest common factor and the lowest common multiple of

17. $2ab^2$, $3a^2b^3$, $4a^4b$. **18.** $15x^3y^2$, $5x^2yz^5$. **19.** $2a^4$, $8a^2b^3c^7$.
20. $57ax^2y$, $76xy^2z^7$. **21.** $32a^4b^3c$, $48a^7bc^5$.
22. $51m^3p^2$, pn, $34mnp^4$. **23.** $49a^4$, $56b^4c$, $21ac^3$.
24. $66a^2b^3cx^4$, $55ab^5xy^3z$, $121x^3yz^7$.

Elementary Fractions.

94. Definition. If a quantity x be divided into b equal parts, and a of these parts be taken, the result is called *the fraction* $\frac{a}{b}$ *of* x. If x be the unit, the fraction $\frac{a}{b}$ of x is called simply "the fraction $\frac{a}{b}$"; so that *the fraction* $\frac{a}{b}$ *represents* a *equal parts,* b *of which make up the unit.*

95. In this chapter we propose to deal only with the easier kinds of fractions, where the numerator and denominator are simple expressions.

Their reduction and simplification will be performed by the usual arithmetical rules. For the proofs of these rules the reader is referred to the *Elementary Algebra for Schools*, Chapter XV.

Rule. *To reduce a fraction to its lowest terms: divide numerator and denominator by every factor which is common to them both, that is by their highest common factor.*

Dividing numerator and denominator of a fraction by a common factor is called *cancelling* that factor.

Examples. (1) $\dfrac{6a^2c}{9ac^2}=\dfrac{2a}{3c}$.

(2) $\dfrac{7x^2yz}{28x^3yz^2}=\dfrac{1}{4xz}$.

(3) $\dfrac{35a^5b^3c}{7ab^2c}=\dfrac{5a^4b}{1}=5a^4b$.

EXAMPLES XII. c.

Reduce to lowest terms :

1. $\dfrac{2a}{4ab}$.
2. $\dfrac{3a^2}{9ab}$.
3. $\dfrac{2bc^2}{6b^2c}$.
4. $\dfrac{2abc}{8a^2bc^2}$.
5. $\dfrac{xy^2z^3}{x^3y^4z}$.
6. $\dfrac{12mn}{15lm}$.
7. $\dfrac{14xy^3}{21x^2z^3}$.
8. $\dfrac{9a^3b}{12ab^3c}$.
9. $\dfrac{15a^2b^2c^3}{18abc^2}$.
10. $\dfrac{5a^3y^2z^4}{15ay^4z}$.
11. $\dfrac{10xy^3}{24x^3y}$.
12. $\dfrac{3m^2n^3p^2}{15mn^2p}$.
13. $\dfrac{15k^2p^3m^4}{25k^3pm^2}$.
14. $\dfrac{27a^4b^3x^2}{45a^3b^4x^4}$.
15. $\dfrac{56a^3c^4z^2}{77ac^2z^3}$.
16. $\dfrac{42x^2y^2z^2}{210x^3y^2z}$

Multiplication and Division of Fractions.

96. **Rule.** *To multiply algebraical fractions: as in Arithmetic, multiply together all the numerators for a new numerator, and all the denominators for a new denominator.*

Example 1. $\frac{2a}{3b} \times \frac{5x^2}{2a^2b} \times \frac{3b^2}{2x} = \frac{2a \times 5x^2 \times 3b^2}{3b \times 2a^2b \times 2x} = \frac{5x}{2a}$,

by cancelling like factors in numerator and denominator.

Example 2. $\frac{3a^2b}{5c^2} \times \frac{7bc}{3a^3} \times \frac{5ca}{7b^2} = 1$,

all the factors cancelling each other.

97. **Rule.** *To divide one fraction by another: invert the divisor and proceed as in multiplication.*

Example.
$$\frac{7a^3}{4x^3y^2} \times \frac{6c^3x}{5ab^2} \div \frac{28a^2c^2}{15b^2xy^2}$$
$$= \frac{7a^3}{4x^3y^2} \times \frac{6c^3x}{5ab^2} \times \frac{15b^2xy^2}{28a^2c^2}$$
$$= \frac{9c}{8x},$$

all the other factors cancelling each other.

EXAMPLES XII. d.

Simplify the following expressions :

1. $\frac{xy}{ab} \times \frac{a^2b^3}{xy^2}$.
2. $\frac{ab}{2cd^3} \times \frac{4c^2d}{ab^3}$.
3. $\frac{2ax^2}{3y^2z} \times \frac{yz^3}{4a^2x}$.
4. $\frac{6a^2x^3}{7ab^2} \times \frac{14b^2c}{12ax}$.
5. $\frac{3ab^2}{5b^3c} \times \frac{15b^2c^2}{9a^2b}$.
6. $\frac{7c^2}{5bc^2} \times \frac{25c^2}{14bc}$.
7. $\frac{a^2m}{b^2y} \times \frac{2cd^2}{3ab} \times \frac{9my}{4m^2}$.
8. $\frac{4a^2b}{9xy} \times \frac{3p^2q^2}{8a^2b^2} \div \frac{pq}{x^2y^2}$.
9. $\frac{2a^3p^2}{5ax^4} \times \frac{10b^2}{4x^2} \div \frac{b^2p^2}{3x^6}$.
10. $\frac{y^2z^3}{zx^3} \times \frac{17y}{x^2z^2} \div \frac{34y^3}{x^5y}$.
11. $\frac{8a^2z^2}{2by} \times \frac{9ax^2}{5a^2z} \times \frac{x^3y^2}{2b^2y}$.
12. $\frac{15b^2}{40c} \times \frac{14d^3}{abc} \div \frac{81d^3}{27c^2}$.

Reduction to a Common Denominator.

98. In order to find the sum or difference of any fractions, we must, as in Arithmetic, first reduce them to a common denominator; and it is most convenient to take the lowest common multiple of the denominators of the given fractions.

Example. Express with lowest common denominator the fractions

$$\frac{a}{3xy},\ \frac{b}{6xyz},\ \frac{c}{2yz}.$$

The lowest common multiple of the denominators is $6xyz$. Multiplying the numerator of each fraction by the factor which is required to make its denominator $6xyz$, we have the equivalent fractions

$$\frac{2az}{6xyz},\ \frac{b}{6xyz},\ \frac{3cx}{6xyz}.$$

Note. The same result would clearly be obtained by dividing the lowest common denominator by each of the denominators in turn, and multiplying the corresponding numerators by the respective quotients.

EXAMPLES XII. e.

Express as equivalent fractions with common denominator:

1. $\frac{x}{2a},\ \frac{2x}{a}.$
2. $\frac{y}{b},\ \frac{2y}{3b}.$
3. $\frac{3a}{2c},\ \frac{4a}{5c}.$
4. $\frac{x}{2y},\ a.$
5. $\frac{m}{4n},\ \frac{3m}{5n}.$
6. $\frac{x}{3y},\ \frac{2a}{x}.$
7. $\frac{a}{b},\ \frac{2a}{b^2}.$
8. $\frac{x}{3y},\ \frac{3x}{y^2}.$
9. $\frac{a}{b},\ \frac{x}{y},\ 1.$
10. $\frac{a}{b},\ \frac{b}{a},\ 2a.$
11. $3,\ \frac{a}{2b},\ \frac{b}{2a}.$
12. $\frac{1}{bc},\ \frac{1}{ca},\ \frac{1}{ab}.$
13. $\frac{a}{2xy},\ \frac{b}{3yz},\ \frac{c}{zx}.$
14. $\frac{m}{n},\ \frac{n}{m},\ \frac{p}{q}.$

Addition and Subtraction of Fractions.

99. **Rule.** *To add or subtract fractions: express all the fractions with their lowest common denominator; form the algebraical sum of the numerators, and retain the common denominator.*

Example 1. Simplify $\frac{5x}{3}+\frac{3}{4}x-\frac{7x}{6}.$

The least common denominator is 12.

$$\text{The expression} = \frac{20x+9x-14x}{12} = \frac{15x}{12} = \frac{5x}{4}.$$

Example 2. Simplify $\frac{3ab}{5x}-\frac{ab}{2x}-\frac{ab}{10x}.$

$$\text{The expression} = \frac{6ab-5ab-ab}{10x} = \frac{0}{10x} = 0.$$

Example 3. Simplify $\frac{2x}{a^2c^2} - \frac{y}{3ca^3}$.

The expression $= \frac{6ax - cy}{3a^3c^2}$, and admits of no further simplification.

Note. The beginner must be careful to distinguish between **erasing equal terms with different signs,** as in Example 2, and **cancelling equal factors** in the course of multiplication, or in reducing fractions to lowest terms. Moreover, in simplifying fractions he must remember that a factor can only be removed from numerator and denominator when it divides each *taken as a whole.*

Thus in $\frac{6ax - cy}{3a^3c^2}$, c cannot be cancelled because it only divides cy and not the *whole* numerator. Similarly a cannot be cancelled because it only divides $6ax$ and not the whole numerator. The fraction is therefore in its simplest form.

When no denominator is expressed the denominator 1 may be understood.

Example 4. $3x - \frac{a^2}{4y} = \frac{3x}{1} - \frac{a^2}{4y} = \frac{12xy - a^2}{4y}$.

If a fraction is not in its lowest terms it should be simplified before combining it with other fractions.

Example 5. $\frac{ax}{2} - \frac{x^2y}{3xy} = \frac{ax}{2} - \frac{x}{3} = \frac{3ax - 2x}{6}$.

EXAMPLES XII. f.

Simplify the following expressions:

1. $\frac{a}{2} + \frac{a}{3}$.
2. $\frac{b}{3} + \frac{b}{4}$.
3. $\frac{x}{4} - \frac{x}{5}$.
4. $\frac{2y}{3} + \frac{y}{6}$.
5. $\frac{a}{5} - \frac{b}{6}$.
6. $\frac{m}{8} - \frac{2n}{20}$.
7. $\frac{p}{7} + \frac{q}{21}$.
8. $\frac{5a}{12} - \frac{b}{4}$.
9. $\frac{a}{x} + \frac{b}{y}$.
10. $\frac{x}{y} - \frac{a}{b}$.
11. $\frac{2a}{3} + \frac{4a}{9b}$.
12. $\frac{ab}{3} - \frac{x^2y}{6xy}$.
13. $\frac{a}{4} - \frac{a}{8} + \frac{a}{12}$.
14. $\frac{2x}{3} - \frac{x}{6} + \frac{9x}{12}$.
15. $\frac{a}{xy} + \frac{2a}{yz} - \frac{3a}{zx}$.
16. $\frac{xy}{5x} - \frac{2y}{3} + \frac{4y}{8}$.
17. $2 + \frac{a}{b} - \frac{b^2}{ab}$.
18. $\frac{a}{p^2} + \frac{b}{pq} - \frac{c}{q^2}$.
19. $a - \frac{x^3}{a^2}$.
20. $\frac{m}{2} - \frac{n^3}{m^2}$.
21. $\frac{a^3}{ab^2} - 3$.
22. $k^4 - \frac{p^6}{k^2}$.
23. $\frac{d^2x}{dy^2} - 2\frac{dx}{dy} + \frac{xy}{2y^2}$.
24. $\frac{a^3}{3a^2b} - \frac{a^3}{ab^2} + \frac{ac}{6bc}$.

CHAPTER XIII.

SIMULTANEOUS EQUATIONS.

100. CONSIDER the equation $2x+5y=23$, which contains *two* unknown quantities.

By transposition we get

$$5y=23-2x;$$

that is, $$y=\frac{23-2x}{5} \qquad\qquad (1).$$

From this it appears that for every value we choose to give to x there will be one corresponding value of y. Thus we shall be able to find as many pairs of values as we please which satisfy the given equation.

For instance, if $x=1$, then from (1) we obtain $y=\frac{21}{5}$.

Again, if $x=-2$, then $y=\frac{27}{5}$; and so on.

But if also we have a second equation containing the same unknown quantities, such as $3x+4y=24$,

we have from this $$y=\frac{24-3x}{4} \qquad\qquad (2).$$

If now we seek values of x and y which satisfy *both* equations, the values of y in (1) and (2) must be identical.

Therefore $$\frac{23-2x}{5}=\frac{24-3x}{4}.$$

Multiplying across $$92-8x=120-15x;$$

$$\therefore \quad 7x=28;$$

$$\therefore \quad x=4.$$

Substituting this value in the first equation, we have

$$8+5y=23;$$

$$\therefore \quad 5y=15;$$

$$\therefore \quad \left.\begin{array}{l} y=3, \\ x=4. \end{array}\right\}$$

and

Thus, if both equations are to be satisfied by the *same* values of x and y, there is only one solution possible.

101. DEFINITION. When two or more equations are satisfied by the *same* values of the unknown quantities they are called **simultaneous equations.**

We proceed to explain the different methods for solving simultaneous equations. In the present chapter we shall confine our attention to the simpler cases in which the unknown quantities are involved in the first degree.

102. In the example already worked we have used the method of solution which best illustrates the meaning of the term *simultaneous equation*; but in practice it will be found that this is rarely the readiest mode of solution. It must be borne in mind that since the two equations are simultaneously true, *any* equation formed by combining them will be satisfied by the values of x and y which satisfy the original equations. Our object will always be to obtain an equation which involves *one only* of the unknown quantities.

103. The process by which we cause either of the unknown quantities to disappear is called **elimination.** We shall consider two methods.

Elimination by Addition or Subtraction.

Example 1. Solve $3x+7y=27$ ....(1),

$5x+2y=16$ ....(2).

To eliminate x we multiply (1) by 5 and (2) by 3, so as to make the coefficients of x in both equations equal. This gives

$$15x+35y=135,$$
$$15x+\ \ 6y=48\ ;$$

subtracting, $$29y=87\ ;$$
$$\therefore\quad y=3.$$

To find x, substitute this value of y in *either* of the given equations.

Thus from (1), $$3x+21=27\ ;$$
$$\therefore\quad x=2,$$
and $$y=3.$$

Note. When one of the unknowns has been found, it is immaterial which of the equations we use to complete the solution. Thus, in the present example, if we substitute 3 for y in (2), we have

$$5x+6=16\ ;$$
$$\therefore\quad x=2, \text{ as before.}$$

Example 2. Solve $7x+2y=47$ .... (1),
$5x-4y=1$ .... (2).

Here it will be more convenient to eliminate y.

Multiplying (1) by 2, $14x+4y=94$,
and from (2) $5x-4y=1$;
adding, $19x=95$;
$\therefore$ $x=5$.

Substitute this value in (1),
$\therefore$ $35+2y=47$;
$\therefore$ $y=6$,

Note. *Add* when the coefficients of one unknown are equal and *unlike* in sign; *subtract* when the coefficients are equal and *like* in sign.

Elimination by Substitution.

Example 3. Solve $2x=5y+1$ .... (1),
$24-7x=3y$.................... (2).

Here we can eliminate x by substituting in (2) its value obtained from (1). Thus

$$24-\frac{7}{2}(5y+1)=3y;$$
$$\therefore\quad 48-35y-7=6y;$$
$$\therefore\quad 41=41y;$$
$$\therefore\quad y=1,$$
and from (1) $x=3$.

104. Any one of the methods given above will be found sufficient; but there are certain arithmetical artifices which will sometimes shorten the work.

Example. Solve $28x-23y=22$ .... (1),
$63x-55y=17$ .... (2).

Noticing that 28 and 63 contain a common factor 7, we shall make the coefficients of x in the two equations equal to the *least common multiple* of 28 and 63 if we multiply (1) by 9 and (2) by 4.

Thus $252x-207y=198$,
$252x-220y=68$;
subtracting, $13y=130$;
that is, $y=10$,
and therefore from (1), $x=9$.

EXAMPLES XIII. a.

Solve the equations:

1. $x+y=19,$ $x-y=7.$
2. $x+y=23,$ $x-y=5.$
3. $x+y=11,$ $x-y=-9.$
4. $x+y=24,$ $x-y=0.$
5. $x-y=6,$ $x+y=0.$
6. $x-y=25,$ $x+y=13.$
7. $3x+5y=50,$ $4x+3y=41.$
8. $x+5y=18,$ $3x+2y=41.$
9. $4x+y=10,$ $5x+7y=47.$
10. $7x-6y=25,$ $5x+4y=51.$
11. $5x+4y=7,$ $4x+5y=2.$
12. $3x-7y=1,$ $4x+y=53.$
13. $7x+5y=45,$ $2x-3y=4.$
14. $4x+5y=4,$ $5x-3y=79.$
15. $11x-7y=43,$ $2x-3y=13.$
16. $4x-3y=0,$ $7x-4y=36.$
17. $2x+3y=22,$ $5x+2y=0.$
18. $7x+3y=65,$ $7x-8y=32.$
19. $13x-y=14,$ $2x-7y=9.$
20. $9x-8y=14,$ $15x-14y=20.$
21. $14x+13y=35,$ $21x+19y=56.$
22. $5x=7y-21,$ $21x-9y=75.$
23. $55x=33y,$ $10x=7y-15.$
24. $5x-7y=11,$ $18x=12y.$
25. $13x-9y=46,$ $11x-12y=17.$
26. $6x-5y=11,$ $28x+21y=7.$
27. $11y-11x=66,$ $7x+8y=3.$
28. $6y-5x=11,$ $4x=7y-22.$
29. $3x+10=5y,$ $7y=4x+13.$
30. $4y=47+3x,$ $5x=30-15y.$
31. $11x+13y=7,$ $13x+11y=17.$
32. $13x-17y=11,$ $29x-39y=17.$
33. $19x+17y=7,$ $41x+37y=17.$

105. We add a few cases in which, before proceeding to solve, it will be necessary to simplify the equations.

Example. Solve
$$5(x+2y)-(3x+11y)=14 \quad\ldots\ldots\ldots (1),$$
$$7x-9y-3(x-4y)=38 \quad\ldots\ldots\ldots (2).$$

From (1),
$$5x+10y-3x-11y=14;$$
$$\therefore\ 2x-y=14 \quad\ldots\ldots\ldots (3).$$

From (2),
$$7x-9y-3x+12y=38;$$
$$\therefore\ 4x+3y=38 \quad\ldots\ldots\ldots (4).$$

From (3),
$$6x-3y=42.$$

By addition, $10x=80$; whence $x=8$. From (3) we obtain $y=2$.

106. Sometimes the value of the second unknown is more easily found by elimination than by substituting the value of the unknown already found.

Example. Solve $$3x - \frac{y-5}{7} = \frac{4x-3}{2} \quad \ldots\ldots (1),$$

$$\frac{3y+4}{5} - \frac{1}{3}(2x-5) = y \quad \ldots\ldots (2).$$

Clear of fractions. Thus

from (1), $$42x - 2y + 10 = 28x - 21;$$

$$\therefore \quad 14x - 2y = -31 \quad \ldots\ldots (3).$$

From (2), $$9y + 12 - 10x + 25 = 15y;$$

$$\therefore \quad 10x + 6y = 37 \quad \ldots\ldots (4).$$

Eliminating y from (3) and (4), we find that

$$x = -\frac{14}{13}.$$

Eliminating x from (3) and (4), we find that

$$y = \frac{207}{26}.$$

107. Simultaneous equations may often be conveniently solved by considering $\frac{1}{x}$ and $\frac{1}{y}$ as the unknown quantities.

Example. Solve $$\frac{8}{x} - \frac{9}{y} = 1 \quad \ldots\ldots (1),$$

$$\frac{10}{x} + \frac{6}{y} = 7 \quad \ldots\ldots (2).$$

Multiply (1) by 2 and (2) by 3: thus

$$\frac{16}{x} - \frac{18}{y} = 2,$$

$$\frac{30}{x} + \frac{18}{y} = 21;$$

adding, $$\frac{46}{x} = 23;$$

multiplying up, $$46 = 23x;$$

$$\therefore \quad x = 2;$$

and by substituting in (1), $$y = 3.$$

EXAMPLES XIII. b.

Solve the equations:

1. $2x - y = 4,\quad \dfrac{x}{2} + \dfrac{y}{4} = 5.$

2. $4x - y = 1,\quad \dfrac{x}{2} + \dfrac{3y}{7} = 4.$

3. $x + 2y = 13,\quad \dfrac{2x}{3} - \dfrac{y}{5} = 1.$

4. $\dfrac{3x}{10} + y = 1,\quad x + 3y = 2.$

5. $x - \dfrac{2y}{3} = 0,\quad 4x - 3y = 1.$

6. $\dfrac{3x}{5} - y = 7,\quad 4x + 5y = 0.$

7. $5x = 4y,\quad \dfrac{4x}{3} - \dfrac{3y}{5} = 7.$

8. $x - y = 0,\quad \dfrac{5}{3}x - \dfrac{9}{2}y = 2\frac{5}{6}.$

9. $x + y = -2,\quad \dfrac{x}{4} + \dfrac{y}{6} = 0.$

10. $\dfrac{1}{2}(x + 3) = 0,\quad \dfrac{1}{6}x - y = 4\frac{1}{2}.$

11. $\dfrac{3}{5}x - 2y = 20,\quad \dfrac{1}{2}(y + 8) = 2.$

12. $\dfrac{1}{3}x - \dfrac{1}{2}y = 0,\quad 3x = 2y.$

13. $3(x - y) + 2(x + y) = 15,\quad 3(x + y) + 2(x - y) = 25.$

14. $3(x + y - 5) = 2(y - x),\quad 3(x - y - 7) + 2(x + y - 2) = 0.$

15. $4(2x - y - 6) = 3(3x - 2y - 5),\quad 2(x - y + 1) + 4x = 3y + 4.$

16. $7(2x - y) + 5(3y - 4x) + 30 = 0,\quad 5(y - x + 3) = 6(y - 2x).$

17. $\dfrac{x + 4}{5} = \dfrac{y - 4}{7} = 2x + y + 4.$

18. $\dfrac{x - 12}{4} = \dfrac{y + 18}{3} = \dfrac{2x + 3y}{2}.$

19. $\dfrac{8}{x} + \dfrac{9}{y} = 7,\quad \dfrac{6}{x} - \dfrac{1}{y} = 2\frac{2}{3}.$

20. $\dfrac{3}{x} + \dfrac{5}{y} = 37,\quad \dfrac{7}{x} - \dfrac{3}{y} = 13.$

21. $\dfrac{10}{x} - \dfrac{3}{y} = 8,\quad \dfrac{3}{x} + \dfrac{2}{y} = -3\frac{2}{5}.$

108. In order to solve simultaneous equations which contain two unknown quantities we have seen that we must have two equations. Similarly we find that in order to solve simultaneous equations which contain three unknown quantities we must have three equations.

Rule. *Eliminate one of the unknowns from any pair of the equations, and then eliminate the same unknown from another pair. Two equations involving two unknowns are thus obtained, which may be solved by the rules already given. The remaining unknown is then found by substituting in any one of the given equations.*

Example. Solve $7x+5y-7z=-8$ .... (1),

$4x+2y-3z=0$ .... (2),

$5x-4y+4z=35$ .... (3).

Choose y as the unknown to be eliminated.

Multiply (2) by 5, $20x+10y-15z=0$;

Multiply (1) by 2, $14x+10y-14z=-16$;

by subtraction, $6x-z=16$ .... (4).

Multiply (2) by 2, $8x+4y-6z=0$;

from (3), $5x-4y+4z=35$;

by addition, $13x-2z=35$.

Multiply (4) by 2, $12x-2z=32$;

by subtraction, $x=3$.

From (4) we find $z=2$,

and from (2), $y=-3$.

109. Some modification of the foregoing rule may often be used with advantage.

Example. Solve $\frac{x}{2}-1=\frac{y}{6}+1=\frac{z}{7}+2$,

$\frac{y}{3}+\frac{z}{2}=13$.

From the equation $\frac{x}{2}-1=\frac{y}{6}+1$,

we have $3x-y=12$ .... (1).

Also from the equation $\frac{x}{2}-1=\frac{z}{7}+2$,

we have $7x-2z=42$ .... (2).

And from the equation $\frac{y}{3}+\frac{z}{2}=13$,

we have $2y+3z=78$ .... (3).

Eliminating z from (2) and (3), we have

$21x+4y=282$;

and from (1) $12x-4y=48$;

whence $x=10$, $y=18$. Also by substitution in (2) we obtain $z=14$.

EXAMPLES XIII. c.

Solve the equations :

1. $3x - 2y + z = 4,\quad 2x + 3y - z = 3,\quad x + y + z = 8.$

2. $3x + 4y - 6z = 16,\quad 4x + y - z = 24,\quad x - 3y - 2z = 1.$

3. $x + 2y + 3z = 32,\quad 4x - 5y + 6z = 27,\quad 7x + 8y - 9z = 14.$

4. $x - y + z = 5,\quad 6x + 3y + 2z = 84,\quad 3x + 4y - 5z = 13.$

5. $7x - 4y - 3z = 0,\quad 5x - 3y + 2z = 12,\quad 3x + 2y - 5z = 0.$

6. $4x + 3y - z = 9,\quad 9x - y + 5z = 16,\quad x + 4y - 3z = 2.$

7. $3y - 6z - 5x = 4,\quad 2z - 3x - y = 8,\quad x - 2y + 2z + 2 = 0.$

8. $3y + 2z + 5x = 21,\quad 8x - 3z + y = 3,\quad 2z + 2x - 3y = 39.$

9. $\frac{1}{2}x + y + \frac{1}{2}z = \frac{1}{2},\quad x + 2y + \frac{1}{3}z = \frac{1}{3},\quad x + y - 9z = 1.$

10. $\frac{1}{2}x - \frac{1}{4}y = 5 - \frac{1}{6}z,\quad \frac{1}{6}x - \frac{1}{3}y = 3 - \frac{1}{6}z,\quad 2y + 7 = \frac{1}{4}(z - x).$

11. $\frac{1}{3}x + \frac{1}{4}(y + z) = 1\frac{2}{3},\quad 4x + \frac{1}{2}(z - y) = 11,\quad \frac{1}{3}(z - 4x) = y.$

12. $2x - \frac{1}{5}(z - 2y) = 2,\quad \frac{1}{3}(x + y) = \frac{1}{7}(3 - z),\quad x = 4y + 3z.$

13. $\dfrac{7 + y - 2z}{3} = y - x = x - z = z - 3.$

14. $\dfrac{x}{3} - \dfrac{y}{2} = y + \dfrac{z}{2} = x + y + z + 2 = 0.$

15. $\dfrac{2x - y - z}{2} = \dfrac{2y - z - x}{3} = \dfrac{2z - x - y}{4} = x - y - z - 6.$

16. $\dfrac{x}{2} + y = 1,\quad \dfrac{y}{3} - z = 3,\quad z + 2y + 3x + 8 = 0.$

CHAPTER XIV.

Problems leading to Simultaneous Equations.

110. In the Examples discussed in the last chapter we have seen that it is essential to have as many equations as there are unknown quantities to determine. Consequently the statement of problems which give rise to simultaneous equations must contain as many independent conditions, or different relations between the unknown quantities, as there are quantities to be determined.

Example 1. Find two numbers whose difference is 11, and one-fifth of whose sum is 9.

Let x be the greater number, y the less;

then $$x-y=11 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Also $$\frac{x+y}{5}=9,$$

or $$x+y=45 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

By addition $2x=56$; and by subtraction $2y=34$.

The numbers are therefore 28 and 17.

Example 2. If 15 lbs. of tea and 10 lbs. of coffee together cost \$15.50, and 25 lbs. of tea and 13 lbs. of coffee together cost \$24.55: find the price of each per pound.

Suppose a pound of tea to cost x cents,

and . . . . . . .coffee. . . . . y. . . . . .

Then from the question we have

$$15x+10y=1550 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$
$$25x+13y=2455 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Multiplying (1) by 5 and (2) by 3, we have

$$75x+50y=7750,$$
$$75x+39y=7365.$$

Subtracting, $$11y=385,$$
$$y=35.$$

And from (1), $$15x+350=1550;$$

whence $$15x=1200;$$
$$\therefore \quad x=80.$$

$\therefore$ the cost of a pound of tea is 80 cents,

and the cost of a pound of coffee is 35 cents.

Example 3. In a bag containing black and white balls, half the number of white is equal to a third of the number of black; and twice the whole number of balls exceeds three times the number of black balls by four. How many balls did the bag contain?

Let x be the number of white balls, and y the number of black balls; then the bag contains $x+y$ balls.

We have the following equations:

$$\frac{x}{2}=\frac{y}{3} \quad \text{.........} \quad (1),$$

$$2(x+y)=3y+4 \quad \text{.........} \quad (2).$$

Substituting from (1) in 2, we obtain

$$\frac{4y}{3}+2y=3y+4;$$

whence $y=12$;

and from (1), $x=8$.

Thus there are 8 white and 12 black balls.

111. In a problem involving the digits of a number the student should carefully notice the way in which the value of a number is algebraically expressed in terms of its digits.

Consider a number of three digits such as 435; its value is $4\times 100+3\times 10+5$. Similarly a number whose digits beginning from the left are x, y, z

$$= x \text{ hundreds} + y \text{ tens} + z \text{ units}$$
$$= 100x+10y+z.$$

Example. A certain number of two digits is three times the sum of its digits, and if 45 be added to it the digits will be reversed; find the number.

Let x be the digit in the tens' place, y the digit in the units' place; then the number will be represented by $10x+y$, and the number formed by reversing the digits will be represented by $10y+x$.

Hence we have the two equations

$$10x+y=3(x+y) \quad \text{.........} \quad (1),$$

and

$$10x+y+45=10y+x \quad \text{.........} \quad (2).$$

From (1), $7x=2y$;

from (2), $y-x=5$.

From these equations we obtain $x=2$, $y=7$.

Thus the number is 27.

EXAMPLES XIV.

1. Find two numbers whose sum is 54, and whose difference is 12.

2. The sum of two numbers is 97 and their difference is 51; find the numbers.

3. One-fifth of the difference of two numbers is 3, and one-third of their sum is 17; find the numbers.

4. One-sixth of the sum of two numbers is 14, and half their difference is 13; find the numbers.

5. Four sheep and seven cows are worth \$131, while three cows and five sheep are worth \$66. What is the value of each animal?

6. A farmer bought 7 horses and 9 cows for \$330. He could have bought 10 horses and 5 cows for the same money; find the price of each animal.

7. Twice *A*'s age exceeds three times *B*'s age by 2 years; if the sum of their ages is 61 years, how old are they?

8. Half of *A*'s age exceeds a quarter of *B*'s age by 1 year, and three-quarters of *B*'s age exceeds *A*'s by 11 years; find the age of each?

9. In eight hours *C* walks 3 miles more than *D* does in six hours, and in seven hours *D* walks 9 miles more than *C* does in six hours; how many miles does each walk per hour?

10. In 9 hours a coach travels one mile more than a train does in 2 hours, but in 3 hours the train travels 2 miles more than the coach does in 13 hours; find the rate of each per hour.

11. A bill of \$15 is paid with half-dollars and quarters, and three times the number of half-dollars exceeds twice the number of quarters by 6; how many of each are used?

12. A bill of \$8.70 is paid with quarters and dimes, and five times the number of dimes exceeds seven times the number of quarters by 6; how many of each are used?

13. Forty-six tons of goods are to be carried in carts and wagons, and it is found that this will require 10 wagons and 14 carts, or else 13 wagons and 9 carts; how many tons can each wagon and each cart carry?

14. A sum of \$11.50 is given to 17 boys and 15 girls; the same amount could have been given to 13 boys and 20 girls; find how much each boy and each girl receives.

15. A certain number of two digits is seven times the sum of the digits, and if 36 be taken from the number the digits will be reversed; find the number.

16. A certain number of two digits is four times the sum of the digits, and if 27 be added to the number the digits will be reversed; find the number.

17. A certain number between 10 and 100 is six times the sum of the digits, and the number exceeds the number formed by reversing the digits by 9; find the number.

18. The digits of a number between 10 and 100 are equal to each other, and the number exceeds 5 times the sum of the digits by 8; find the number.

19. A man has $380 in silver dollars, half-dollars, and quarters; the number of the coins is 852, and their weight is 235 ounces. If a dollar weighs $\frac{1}{2}$ oz., a half-dollar $\frac{1}{4}$ oz., and a quarter $\frac{1}{5}$ oz., find how many of each kind of the coins he has.

20. A man has $22 worth of silver in half-dollars, quarters, and dimes. He has in all 70 coins. If he changed the half-dollars for dimes and the quarters for half-dollars, he would then have 180 coins. How many of each had he at first?

21. Divide $100 between 3 men, 5 women, 4 boys, and 3 girls, so that each man shall have as much as a woman and a girl, each woman as much as a boy and a girl, and each boy half as much as a man and a girl.

22. If 17 lbs. of sugar and 5 lbs. of coffee cost $2.50, and 10 lbs. of sugar and 10 lbs. of coffee cost $3.80, find the cost per lb. of sugar and coffee.

23. The value of a number of coins consisting of dollars and half-dollars amounts to $22.50; the number of dollars exceeds five times the number of half-dollars by 6. Find the number of each.

24. A sum of $23.80 is divided among 11 men and 16 women; the same sum could have been divided among 19 men and 6 women. Find how much each man and woman receives.

25. Two articles A and B are sold for 20 cents and 30 cents per lb. respectively; a person spends $6.50 in buying such articles. If he had bought half as much again of A and one-third as much again of B, he would have spent $9.00. What weight of each did he buy?

CHAPTER XV

INVOLUTION.

112. DEFINITION. **Involution** is the general name for multiplying an expression by itself so as to find its second, third fourth, or any other power.

Involution may always be effected by actual multiplication. Here, however, we shall give some rules for writing down at once

(1) any power of a simple expression ;

(2) the square and cube of any binomial ;

(3) the square of any multinomial.

113. It is evident from the Rule of Signs that

(1) no *even* power of *any* quantity can be *negative ;*

(2) any *odd* power of a quantity will have *the same sign* as the quantity itself.

Note. It is especially worthy of remark that the *square* of every expression, whether positive or negative, is *positive.*

114. From definition we have, by the rules of multiplication.

$$(a^2)^3 = a^2 . a^2 . a^2 = a^{2+2+2} = a^6.$$

$$(-x^3)^2 = (-x^3)(-x^3) = x^{3+3} = x^6.$$

$$(-a^5)^3 = (-a^5)(-a^5)(-a^5) = -a^{5+5+5} = -a^{15}.$$

$$(-3a^3)^4 = (-3)^4(a^3)^4 = 81a^{12}.$$

Hence we obtain a rule for raising a simple expression to any proposed power.

Rule. (1) *Raise the coefficient to the required power by Arithmetic, and prefix the proper sign found by the Rule of Signs.*

(2) *Multiply the index of every factor of the expression by the exponent of the power required.*

Examples.

$$(-2x^3)^5 = -32x^{15}.$$

$$(-3ab^3)^4 = 81a^4b^{12}.$$

$$\left(\frac{2ab^3}{3x^2y}\right)^4 = \frac{16a^4b^{12}}{81x^8y^4}.$$

It will be seen that in the last case *the numerator and the denominator are operated upon separately.*

EXAMPLES XV. a.

Write down the square of each of the following expressions :

1. a^2b. 2. $3ac^3$. 3. $5xy^2$. 4. $6b^3c^2$.

5. $4a^2bc^3$. 6. $-3x^2y^5$. 7. $-2a^2b^3c$. 8. $-3dx^4$.

9. $\frac{a^2c}{bd^3}$. 10. $\frac{2x^3}{y^2}$. 11. $\frac{3a^3}{4b^2}$. 12. $-\frac{5}{7xy^2}$.

13. $-\frac{8p^2q^5}{3}$. 14. $\frac{m^2n^6}{9xy^4}$. 15. $-\frac{1}{4x^2yz^3}$. 16. $-\frac{3pq^2r^3}{5a^2x}$.

Write down the cube of each of the following expressions :

17. $2x$. 18. $3ab^2$. 19. $4x^3$. 20. $-3a^2b$.

21. $-4x^3y^2$. 22. $-b^2cd^3$. 23. $-6y^4$. 24. $-4p^3q^5$.

25. $\frac{1}{p^4r^3}$. 26. $-\frac{2}{ab^2c^3}$. 27. $-\frac{3x^2}{4yz^3}$. 28. $-\frac{2}{3}x^3$.

Write down the value of each of the following expressions :

29. $(ab^2)^4$. 30. $(-x^2y)^5$. 31. $(-2m^2n^3)^6$. 32. $(-x^3y^2)^7$.

33. $\left(\frac{1}{3a^2}\right)^5$. 34. $\left(-\frac{2a^3x}{3by^2}\right)^4$. 35. $\left(-\frac{a^3}{3}\right)^7$. 36. $\left(-\frac{2}{3}a^2\right)^6$.

To Square a Binomial.

115. By multiplication we have

$$(a+b)^2 = (a+b)(a+b)$$
$$= a^2 + 2ab + b^2 \quad\text{.................. (1).}$$
$$(a-b)^2 = (a-b)(a-b)$$
$$= a^2 - 2ab + b^2 \quad\text{.................. (2).}$$

These formulæ may be enunciated verbally as follows:

Rule 1. *The square of the sum of two quantities is equal to the sum of their squares increased by twice their product.*

Rule 2. *The square of the difference of two quantities is equal to the sum of their squares diminished by twice their product.*

Example 1. $(x+2y)^2 = x^2+2 \cdot x \cdot 2y+(2y)^2$
$= x^2+4xy+4y^2.$

Example 2. $(2a^3-3b^2)^2=(2a^3)^2-2 \cdot 2a^3 \cdot 3b^2+(3b^2)^2$
$=4a^6-12a^3b^2+9b^4.$

To Square a Multinomial.

116. By the preceding article

$$\begin{aligned}(a+b+c)^2&=\{(a+b)+c\}^2\\ &=(a+b)^2+2(a+b)c+c^2\\ &=a^2+b^2+c^2+2ab+2ac+2bc.\end{aligned}$$

In the same way we may prove

$$(a-b+c)^2=a^2+b^2+c^2-2ab+2ac-2bc.$$
$$(a+b+c+d)^2=a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd.$$

In each of these instances we observe that the square consists of

(1) the sum of the squares of the several terms of the given expression;

(2) twice the sum of the products two and two of the several terms, taken with their proper signs; that is, in each product the sign is + or − according as the quantities composing it have like or unlike signs.

Note. The *square terms* are always positive.

The same laws hold whatever be the number of terms in the expression to be squared.

Rule. *To find the square of any multinomial: to the sum of the squares of the several terms add twice the product (with the proper sign) of each term into each of the terms that follow it.*

Ex. 1. $(x-2y-3z)^2=x^2+4y^2+9z^2-2 \cdot x \cdot 2y-2 \cdot x \cdot 3z+2 \cdot 2y \cdot 3z$
$=x^2+4y^2+9z^2-4xy-6xz+12yz.$

Ex. 2. $(1+2x-3x^2)^2=1+4x^2+9x^4+2 \cdot 1 \cdot 2x-2 \cdot 1 \cdot 3x^2-2 \cdot 2x \cdot 3x^2$
$=1+4x^2+9x^4+4x-6x^2-12x^3$
$=1+4x-2x^2-12x^3+9x^4,$

by collecting like terms and rearranging.

EXAMPLES XV. b.

Write down the square of each of the following expressions :

1. $x+2y$.	2. $x-2y$.	3. $a+3b$.	4. $2a-3b$.
5. $3a+b$.	6. $x-5y$.	7. $2m+7n$.	8. $9-x$.
9. $2-ab$.	10. $abc+1$.	11. $ab-cd$.	12. $2ab+xy$.
13. $1-x^2$.	14. $3+2pq$.	15. x^2-3x.	16. $2a+ab$.

17. $a+b-c$.	18. $a-b-c$.	19. $2a+b+c$.
20. $2x-y-z$.	21. $x+3y-2z$.	22. x^2+x+1.
23. $3x+2p-q$.	24. $1-2x-3x^2$.	25. $2-3x+x^2$.
26. $x+y+a-b$.	27. $m-n+p-q$.	28. $2a+3b+x-2y$.

To Cube a Binomial.

117. By actual multiplication, we have

$$(a+b)^3=(a+b)(a+b)(a+b)$$
$$=a^3+3a^2b+3ab^2+b^3.$$

Also $$(a-b)^3=a^3-3a^2b+3ab^2-b^3.$$

By observing the law of formation of the terms in these results we can write down the cube of any binomial.

Example 1. $$(2x+y)^3=(2x)^3+3(2x)^2y+3(2x)y^2+y^3$$
$$=8x^3+12x^2y+6xy^2+y^3.$$

Example 2. $$(3x-2a^2)^3=(3x)^3-3(3x)^2(2a^2)+3(3x)(2a^2)^2-(2a^2)^3$$
$$=27x^3-54x^2a^2+36xa^4-8a^6.$$

EXAMPLES XV. c.

Write down the cube of each of the following expressions :

1. $p+q$.	2. $m-n$.	3. $a-2b$.	4. $2c+d$.
5. $x+3y$.	6. $x+yz$.	7. $2xy-1$	8. $5a+2$.
9. x^2-1.	10. $2x^2+y^2$.	11. $2a^3-3b^3$.	12. $4a^2-3$

CHAPTER XVI.

EVOLUTION.

118. DEFINITION. The **root** of any proposed expression is that quantity which being multiplied by itself the requisite number of times produces the given expression.

The operation of finding the root is called **Evolution**: it is the reverse of Involution.

119. By the Rule of Signs we see that

(1) any *even* root of a *positive* quantity may be either *positive* or *negative;*

(2) *no negative* quantity can have an *even* root;

(3) every *odd* root of a quantity has the same sign as the quantity itself.

Note. It is especially worthy of remark that every positive quantity has two square roots equal in magnitude, but opposite in sign.

Example. $$\sqrt{9a^2x^6} = \pm 3ax^3.$$

In the present chapter, however, we shall confine our attention to the positive root.

Examples.

$$\sqrt{a^6b^4} = a^3b^2, \text{ because } (a^3b^2)^2 = a^6b^4.$$

$$\sqrt[3]{-x^9} = -x^3, \text{ because } (-x^3)^3 = -x^9.$$

$$\sqrt[5]{c^{20}} = c^4, \text{ because } (c^4)^5 = c^{20}.$$

$$\sqrt[4]{81x^{12}} = 3x^3, \text{ because } (3x^3)^4 = 81x^{12}.$$

120. From the foregoing examples we may deduce a general rule for extracting any proposed root of a simple expression:

Rule. (1) *Find the root of the coefficient by Arithmetic, and prefix the proper sign.*

(2) *Divide the exponent of every factor of the expression by the index of the proposed root.*

Examples.

$$\sqrt[3]{-64x^6} = -4x^2.$$

$$\sqrt[4]{16a^8} = 2a^2.$$

$$\sqrt{\frac{81x^{10}}{25c^4}} = \frac{9x^5}{5c^2}.$$

EXAMPLES XVI. a.

Write down the square root of each of the following expressions:

1. $9x^4y^2$. 2. $25a^6b^4$. 3. $49c^2d^6$. 4. $a^6b^2c^{16}$.

5. $36x^6y^{36}$. 6. $16x^8$. 7. $x^4y^6z^2$. 8. $9p^6q^{12}$.

9. $\dfrac{4x^6}{16a^4}$. 10. $\dfrac{a^{36}}{36}$. 11. $\dfrac{16x^{64}}{25}$. 12. $\dfrac{144}{a^{12}}$.

Write down the cube root of each of the following expressions:

13. x^6y^9. 14. $-a^6b^3$. 15. $8x^{27}$. 16. $-27x^9$.

17. $-\dfrac{b^{27}}{27}$. 18. $\dfrac{8a^9b^{12}}{y^{15}}$. 19. $\dfrac{125a^9x^{21}}{27c^6}$. 20. $-\dfrac{64a^{27}b^3}{x^9}$.

Write down the value of each of the following expressions:

21. $\sqrt[4]{x^4y^{12}}$. 22. $\sqrt[6]{a^{24}x^{18}}$. 23. $\sqrt[5]{-x^{20}y^{30}}$.

24. $\sqrt[6]{64a^{42}}$. 25. $\sqrt[7]{a^{21}b^{14}}$. 26. $\sqrt[9]{p^9q^{27}}$.

27. $\sqrt[7]{-x^{35}y^{56}}$. 28. $\sqrt[4]{81x^4y^{12}}$. 29. $\sqrt[5]{32a^5b^{10}c^{25}}$.

121. By the formulæ in Art. 115 we are able to write down the square of any binomial.

Thus $(2x+3y)^2=4x^2+12xy+9y^2$.

Conversely, by observing the form of the terms of an expression, it may sometimes be recognised as a complete square, and its square root written down at once.

Example 1. Find the square root of $25x^2-40xy+16y^2$.

$$\begin{aligned}\text{The expression} &= (5x)^2-2\,.\,20xy+(4y)^2\\ &= (5x)^2-2(5x)(4y)+(4y)^2\\ &= (5x-4y)^2.\end{aligned}$$

Thus the required square root is $5x-4y$.

Example 2. Find the square root of $\dfrac{64a^2}{9b^2}+4+\dfrac{32a}{3b}$.

$$\begin{aligned}\text{The expression} &= \left(\frac{8a}{3b}\right)^2+(2)^2+2\left(\frac{16a}{3b}\right)\\ &= \left(\frac{8a}{3b}\right)^2+2\left(\frac{8a}{3b}\right)(2)+(2)^2\\ &= \left(\frac{8a}{3b}+2\right)^2.\end{aligned}$$

Thus the required square root is $\dfrac{8a}{3b}+2$.

122. When the square root cannot be easily determined by inspection we must have recourse to the rule explained in the next article, which is quite general, and applicable to all cases. *But the student is advised, here and elsewhere, to employ methods of inspection in preference to rules.*

To Find the Square Root of a Compound Expression.

123. Since the square of $a+b$ is $a^2+2ab+b^2$, we have to discover a process by which a and b, the terms of the root, can be found when $a^2+2ab+b^2$ is given.

The first term, a, is the square root of a^2.

Arrange the terms according to powers of one letter a. The first term is a^2, and its square root is a. Set this down as the first term of the required root. Subtract a^2 from the given expression and the remainder is $2ab+b^2$ or $(2a+b)\times b$.

Now the first term $2ab$ of the remainder is the product of $2a$ and b. Thus to obtain b we divide the first term of the remainder by the double of the term already found; if we add this new term to $2a$ we obtain the complete divisor $2a+b$.

The work may be arranged as follows:

$$\begin{array}{lr} & a^2+2ab+b^2\,(a+b \\ & a^2 \qquad\qquad \\ 2a+b & 2ab+b^2 \\ & 2ab+b^2 \end{array}$$

Example. Find the square root of $9x^2-42xy+49y^2$.

$$\begin{array}{lr} & 9x^2-42xy+49y^2\,(3x-7y \\ & 9x^2 \qquad\qquad\qquad \\ 6x-7y & -42xy+49y^2 \\ & -42xy+49y^2 \end{array}$$

Explanation. The square root of $9x^2$ is $3x$, and this is the first term of the root.

By doubling this we obtain $6x$, which is the first term of the divisor. Divide $-42xy$, the first term of the remainder, by $6x$ and we get $-7y$, the new term in the root, which has to be annexed both to the root and divisor. Next multiply the complete divisor by $-7y$ and subtract the result from the first remainder. There is now no remainder and the root has been found.

124. The rule can be extended so as to find the square root of any multinomial. The first two terms of the root will be obtained as before. When we have brought down *the second remainder*, the first part of the new divisor is obtained by doubling the terms of the root already found. We then divide the first term of the remainder by the first term of the new divisor, and set down the result as the next term in the root and in the divisor. We next multiply the complete divisor by the last term of the root and subtract the product from the last remainder. If there is now no remainder the root has been found; if there is a remainder we continue the process.

Example. Find the square root of

$$25x^2a^2 - 12xa^3 + 16x^4 + 4a^4 - 24x^3a.$$

Rearrange in descending powers of x.

$$\begin{array}{l|l}
 & 16x^4 - 24x^3a + 25x^2a^2 - 12xa^3 + 4a^4 \;(\; 4x^2 - 3xa + 2a^2 \\
 & 16x^4 \\ \hline
8x^2 - 3xa & -24x^3a + 25x^2a^2 \\
 & -24x^3a + 9x^2a^2 \\ \hline
8x^2 - 6xa + 2a^2 & \qquad\qquad 16x^2a^2 - 12xa^3 + 4a^4 \\
 & \qquad\qquad 16x^2a^2 - 12xa^3 + 4a^4 \\ \hline
\end{array}$$

Explanation. When we have obtained two terms in the root, $4x^2 - 3xa$, we have a remainder

$$16x^2a^2 - 12xa^3 + 4a^4.$$

Double the terms of the root already found and place the result, $8x^2 - 6xa$, as the first part of the divisor. Divide $16x^2a^2$, the first term of the remainder, by $8x^2$, the first term of the divisor; we get $+2a^2$ which we annex both to the root and divisor. Now multiply the complete divisor by $2a^2$ and subtract. There is no remainder and the root is found.

125. Sometimes the following method may be used.

Example. Find by inspection the square root of

$$4a^2 + b^2 + c^2 + 4ab - 4ac - 2bc.$$

Arrange the terms in descending powers of a, and let the other letters be arranged alphabetically; then

$$\begin{aligned}
\text{the expression} &= 4a^2 + 4ab - 4ac + b^2 - 2bc + c^2 \\
&= 4a^2 + 4a(b - c) + (b - c)^2 \\
&= (2a)^2 + 2 \,.\, 2a(b - c) + (b - c)^2;
\end{aligned}$$

whence the square root is $2a + (b - c)$. [Art. 121.]

EXAMPLES XVI. b.

By inspection or otherwise, find the square root of each of the following expressions:

1. $a^2 - 8a + 16$.
2. $x^2 + 14x + 49$.
3. $64 + 48x + 9x^2$.
4. $25 - 30m + 9m^2$.
5. $36n^4 - 84n^2 + 49$.
6. $81 + 144y^3 + 64y^6$.
7. $x^6 - 6x^3y^4z^4 + 9y^8z^8$.
8. $4a^2b^4 - 12ab^2c^5 + 9c^{10}$.
9. $\frac{1}{4}x^2 - 3xy^3 + 9y^6$.
10. $\frac{9a^2}{b^2} + \frac{24ac}{bd} + \frac{16c^2}{d^2}$.
11. $\frac{9a^2}{25b^2} - 2 + \frac{25b^2}{9a^2}$.
12. $\frac{16x^4}{49y^2} + \frac{49y^4}{16x^2} + 2xy$.
13. $16x^4 - 32x^3 + 24x^2 - 8x + 1$.
14. $25 - 30a + 29a^2 - 12a^3 + 4a^4$.
15. $9a^8 - 12a^6 - 2a^4 + 4a^2 + 1$.
16. $25p^4 - 30p^3 + 121 - 101p^2 + 66p$.
17. $8x^3 + 1 + 4x^4 - 4x$.
18. $201a^2 - 108a^3 + 100 + 36a^4 - 180a$.
19. $a^2 + b^2 + c^2 + 2ab - 2ac - 2bc$.
20. $y^2z^2 + z^2x^2 + x^2y^2 - 2x^2yz + 2xy^2z - 2xyz^2$.
21. $a^4 - 2a^3 + \frac{3a^2}{2} - \frac{a}{2} + \frac{1}{16}$.
22. $\frac{a^2}{9} - \frac{4ax}{3} + \frac{x^4}{4} + 4x^2 + \frac{ax^2}{3} - 2x^3$.
23. $9m^4 + \frac{n^2}{4} + \frac{9}{16} + \frac{3n}{4} + \frac{9m^2}{2} + 3m^2n$.
24. $9x^4 + 144x^2 + 12ax^2 + 4a^2 - 72x^3 - 48ax$.
25. $x^8 - 4x^7 + 4x^6 + 6x^5 - 14x^4 + 4x^3 + 9x^2 - 6x + 1$.
26. $a^2 + 9b^2 + c^2 - 6ab + 6bc - 2ac$.
27. $\frac{m^4}{n^4} - 2\frac{m^3}{n^3} + 5\frac{m^2}{n^2} - 4\frac{m}{n} + 4$.
28. $\frac{9a^2}{b^2} - 5 + \frac{b^2}{a^2} - \frac{6a}{b} + \frac{2b}{a}$.

[If preferred, the remainder of this chapter may be postponed and taken at a later stage.]

To Find the Cube Root of a Compound Expression.

126. Since the cube of $a+b$ is $a^3+3a^2b+3ab^2+b^3$, we have to discover a process by which a and b, the terms of the root, can be found when $a^3+3a^2b+3ab^2+b^3$ is given.

The first term a is the cube root of a^3.

Arrange the terms according to powers of one letter a; then the first term is a^3, and its cube root a. Set this down as the first term of the required root. Subtract a^3 from the given expression and the remainder is

$$3a^2b+3ab^2+b^3 \text{ or } (3a^2+3ab+b^2)\times b.$$

Now the first term of the remainder is the product of $3a^2$ and b. Thus to obtain b we divide the first term of the remainder by three times the square of the term already found.

Having found b we can complete the divisor, which consists of the following three terms:

1. Three times the square of a, the term of the root already found.

2. Three times the product of this first term a, and the new term b.

3. The square of b.

The work may be arranged as follows:

$$\begin{array}{lll}
 & & a^3+3a^2b+3ab^2+b^3\,(\,a+b \\
 & & a^3 \\
3(a)^2 = 3a^2 & & 3a^2b+3ab^2+b^3 \\
3\times a\times b = \quad +3ab & & \\
(b)^2 = \qquad\quad +b^2 & & \\
\qquad 3a^2+3ab+b^2 & & 3a^2b+3ab^2+b^3
\end{array}$$

mple 1. Find the cube root of $8x^3-36x^2y+54xy^2-27y^3$.

Arra…
letters be
the expres…

$$\begin{array}{lll}
 & & 8x^3-36x^2y+54xy^2-27y^3\,(\,2x-3y \\
 & & 8x^3 \\
= 12x^2 & & -36x^2y+54xy^2-27y^3 \\
) = \qquad -18xy & & \\
\qquad\qquad +9y^2 & & \\
{}^2-18xy+9y^2 & & -36x^2y+54xy^2-27y^3
\end{array}$$

whence the square

Example 2. Find the cube root of $8x^6 - 48x^5 + 60x^4 + 80x^3 - 90x^2 - 108x - 27$.

$$8x^6 - 48x^5 + 60x^4 + 80x^3 - 90x^2 - 108x - 27 \;(\; \mathbf{2x^2 - 4x - 3}$$

$$8x^6$$

$$\begin{array}{rl}
3 \times (2x^2)^2 = 12x^4 & \\
3 \times (2x^2) \times (-4x) = \quad -24x^3 & \\
(-4x)^2 = \quad\quad +16x^2 & -48x^5 + 60x^4 + 80x^3 \\
\hline
12x^4 - 24x^3 + 16x^2 & -48x^5 + 96x^4 - 64x^3 \\
\hline
3 \times (2x^2 - 4x)^2 = 12x^4 - 48x^3 + 48x^2 & -36x^4 + 144x^3 - 90x^2 - 108x - 27 \\
3 \times (2x^2 - 4x) \times (-3) = \quad -18x^2 + 36x & \\
(-3)^2 = \quad\quad +9 & \\
\hline
12x^4 - 48x^3 + 30x^2 + 36x + 9 & -36x^4 + 144x^3 - 90x^2 - 108x - 27 \\
\end{array}$$

Explanation. When we have obtained two terms in the root, $2x^2 - 4x$, we have a remainder

$$-36x^4 + 144x^3 - 90x^2 - 108x - 27.$$

Take 3 times the square of the root already found and place the result, $12x^4 - 48x^3 + 48x^2$, as the first part of the new divisor. Divide $-36x^4$, the first term of the remainder, by $12x^4$, the first term of the divisor; this gives -3 a new term of the root. To complete the divisor we take 3 times the product of $2x^2 - 4x$ and -3, and also the square of -3. Now multiply the complete divisor by -3 and subtract; there is no remainder and the root is found.

EXAMPLES XVI. c.

Find the cube root of each of the following expressions:

1. $a^3+12a^2+48a+64$.
2. $8x^3+12x^2+6x+1$.
3. $64x^3-144x^2+108x-27$.
4. $8p^6-36p^4+54p^2-27$.
5. $m^3-18m^2+108m-216$.
6. $x^6+6x^4y^2+12x^2y^4+8y^6$.
7. $1-3c+6c^2-7c^3+6c^4-3c^5+c^6$.
8. $8+36m+66m^2+63m^3+33m^4+9m^5+m^6$.
9. $216-108k+342k^2-109k^3+171k^4-27k^5+27k^6$.
10. $48y^5+108y+60y^4-90y^2-27+8y^6-80y^3$.
11. $64+192k+240k^2+160k^3+60k^4+12k^5+k^6$.
12. $x^3-6x^2y-3x^2z+12xy^2+12xyz+3xz^2-8y^3-12y^2z-6yz^2-z^3$.

[For additional examples see *Elementary Algebra.*]

127. The ordinary rules for extracting square and cube roots in Arithmetic are based upon the algebraical methods explained in the present chapter. The following example is given to illustrate the arithmetical process.

Example. Find the cube root of 614125.

Since 614125 lies between 512000 and 729000, that is between $(80)^3$ and $(90)^3$, its cube root lies between 80 and 90 and therefore consists of two figures.

$$\begin{array}{rrr|l}
 & & & \quad a+b \\
 & & & 614125\ (\,80+5=85 \\
 & & & 512000 \\
\hline
3a^2= & 3\times(80)^2= & 19200 & 102125 \\
3\times a\times b= & 3\times 80\times 5= & 1200 & \\
b^2= & 5\times 5= & 25 & \\
\cline{3-3}
 & & 20425 & 102125 \\
\end{array}$$

In Arithmetic the ciphers are usually omitted, and there are other modifications of the algebraical rules.

CHAPTER XVII.

Resolution into Factors.

128. Definition. When an algebraical expression is the product of two or more expressions each of these latter quantities is called a **factor** of it, and the determination of these quantities is called the **resolution** of the expression into its factors.

In this chapter we shall explain the principal rules by which the resolution of expressions into their component factors may be effected.

Expressions in which Each Term is divisible by a Common Factor.

129. Such expressions may be simplified by dividing each term separately by this factor, and enclosing the quotient within brackets; the common factor being placed outside as a coefficient.

Example 1. The terms of the expression $3a^2 - 6ab$ have a common factor $3a$;

$$\therefore \quad 3a^2 - 6ab = 3a(a - 2b).$$

Example 2. $5a^2bx^3 - 15abx^2 - 20b^3x^2 = 5bx^2(a^2x - 3a - 4b^2).$

EXAMPLES XVII. a.

Resolve into factors:

1. $x^2 + ax.$ **2.** $2a^2 - 3a.$ **3.** $a^3 - a^2.$ **4.** $a^3 - a^2b.$

5. $3m^2 - 6mn.$ **6.** $p^2 + 2p^2q.$ **7.** $x^5 - 5x^2.$ **8.** $y^2 + xy.$

9. $5a^2 - 25a^2b.$ **10.** $12x + 48x^2y.$ **11.** $10c^3 - 25c^4d.$

12. $27 - 162x.$ **13.** $x^2y^2z^2 + 3xy.$ **14.** $17x^2 - 51x.$

15. $2a^3 - a^2 + a.$ **16.** $3x^3 + 6a^2x^2 - 3a^3x.$

17. $7p^2 - 7p^3 + 14p^4.$ **18.** $4b^5 + 6a^2b^3 - 2b^2.$

19. $x^3y^3 - x^2y^2 + 2xy.$ **20.** $26a^3b^5 + 39a^4b^2.$

Expressions in which the Terms can be so grouped as to contain a Compound Factor that is Common.

130. The method is shown in the following examples.

Example 1. Resolve into factors $x^2-ax+bx-ab$.

Since the first two terms contain a common factor x, and the last two terms a common factor b, we have

$$\begin{aligned} x^2-ax+bx-ab &= (x^2-ax)+(bx-ab) \\ &= x(x-a)+b(x-a) \\ &= (x-a) \text{ taken } x \text{ times } \textit{plus } (x-a) \text{ taken } b \text{ times} \\ &= (x-a) \text{ taken } (x+b) \text{ times} \\ &= (x-a)(x+b). \end{aligned}$$

Example 2. Resolve into factors $6x^2-9ax+4bx-6ab$.

$$\begin{aligned} 6x^2-9ax+4bx-6ab &= (6x^2-9ax)+(4bx-6ab) \\ &= 3x(2x-3a)+2b(2x-3a) \\ &= (2x-3a)(3x+2b). \end{aligned}$$

Example 3. Resolve into factors $12a^2+bx^2-4ab-3ax^2$.

$$\begin{aligned} 12a^2+bx^2-4ab-3ax^2 &= (12a^2-4ab)-(3ax^2-bx^2) \\ &= 4a(3a-b)-x^2(3a-b) \\ &= (3a-b)(4a-x^2). \end{aligned}$$

EXAMPLES XVII. b.

Resolve into factors:

1. $x^2+xy+xz+yz$.
2. $x^2-xz+xy-yz$.
3. $a^2+2a+ab+2b$.
4. $a^2+ac+4a+4c$.
5. $2a+2x+ax+x^2$.
6. $3q-3p+pq-p^2$.
7. $am-bm-an+bn$.
8. $ab-by-ay+y^2$.
9. $pq+qr-pr-r^2$.
10. $2mx+nx+2my+ny$.
11. $ax-2ay-bx+2by$.
12. $2a^2+3ab-2ac-3bc$.
13. ac^2+b+bc^2+a.
14. $ac^2-2a-bc^2+2b$.
15. a^3-a^2+a-1.
16. $2x^3+3+2x+3x^2$.
17. $a^2x-aby+2ax-2by$.
18. $axy+bcxy-az-bcz$.

Trinomial Expressions.

131. In Chap. v. Art. 48 attention has been drawn to the way in which, in forming the product of two binomials, the coefficients of the different terms combine so as to give a trinomial result.

Thus
$$(x+5)(x+3) = x^2+8x+15 \ldots\ldots\ldots(1),$$
$$(x-5)(x-3) = x^2-8x+15 \ldots\ldots\ldots(2),$$
$$(x+5)(x-3) = x^2+2x-15 \ldots\ldots\ldots(3),$$
$$(x-5)(x+3) = x^2-2x-15 \ldots\ldots\ldots(4).$$

We now propose to consider the converse problem: namely, the resolution of a trinomial expression, similar to those which occur on the right-hand side of the above identities, into its component binomial factors.

By examining the above results, we notice that:

1. The first term of both the factors is x.

2. The product of the second terms of the two factors is equal to the third term of the trinomial; e.g. in (2) above we see that 15 is the product of -5 and -3; and in (3) we see that -15 is the product of $+5$ and -3.

3. The algebraic sum of the second terms of the two factors is equal to the coefficient of x in the trinomial; e.g. in (4) the sum of -5 and $+3$ gives -2, the coefficient of x in the trinomial.

The application of these laws will be easily understood from the following examples.

Example 1. Resolve into factors $x^2+11x+24$.

The second terms of the factors must be such that their product is $+24$, and their sum $+11$. It is clear that they must be $+8$ and $+3$.

$$\therefore \quad x^2+11x+24 = (x+8)(x+3).$$

Example 2. Resolve into factors $x^2-10x+24$.

The second terms of the factors must be such that their product is $+24$, and their sum -10. Hence they must *both* be *negative*, and it is easy to see that they must be -6 and -4.

$$\therefore \quad x^2-10x+24 = (x-6)(x-4).$$

Example 3.
$$x^2-18x+81 = (x-9)(x-9) = (x-9)^2.$$

Example 4.
$$x^4+10x^2+25 = (x^2+5)(x^2+5) = (x^2+5)^2.$$

Example 5. Resolve into factors $x^2-11ax+10a^2$.

The second terms of the factors must be such that their product is $+10a^2$, and their sum $-11a$. Hence they must be $-10a$ and $-a$.

$$\therefore \quad x^2-11ax+10a^2 = (x-10a)(x-a).$$

Note. In examples of this kind the student should always verify his results, by forming the product (*mentally*, as explained in Chap. V.) of the factors he has chosen.

EXAMPLES XVII. c.

Resolve into factors:

1. x^2+3x+2.	2. y^2+5y+6.	3. $y^2+7y+12$.
4. a^2-3a+2.	5. a^2-6a+8.	6. b^2-5b+6.
7. $b^2+13b+42$.	8. $b^2-13b+40$.	9. $z^2-13z+36$.
10. $x^2-15x+56$.	11. $x^2-15x+54$.	12. $z^2+15z+44$.
13. $b^2-12b+36$.	14. $a^2+15a+56$.	15. $a^2-12a+27$.
16. $x^2+9x+20$.	17. $x^2-10x+9$.	18. $x^2-16x+64$.
19. $y^2-23y+102$.	20. $y^2-24y+95$.	21. $y^2+54y+729$.
22. $a^2+10ab+21b^2$.	23. $a^2+12ab+11b^2$.	24. $a^2-23ab+132b^2$.
25. m^4+8m^2+7.	26. $m^4+9m^2n^2+14n^4$.	27. $6-5x+x^2$.
28. $54-15a+a^2$.	29. $13+14y+y^2$.	30. $216-35a+a^2$.

132. Next consider a case where the third term of the trinomial is negative.

Example 1. Resolve into factors $x^2+2x-35$.

The second terms of the factors must be such that their product is -35, and their *algebraical sum* $+2$. Hence they must have *opposite* signs, and the greater of them must be *positive* in order to give its sign to their sum.

The required terms are therefore $+7$ and -5.

$$\therefore \quad x^2+2x-35=(x+7)(x-5).$$

Example 2. Resolve into factors $x^2-3x-54$.

The second terms of the factors must be such that their product is -54, and their *algebraical sum* -3. Hence they must have *opposite* signs, and the greater of them must be *negative* in order to give its sign to their sum.

The required terms are therefore -9 and $+6$.

$$\therefore \quad x^2-3x-54=(x-9)(x+6).$$

Remembering that in these cases the numerical quantities *must have opposite signs*, if preferred, the following method may be adopted.

Example 3. Resolve into factors $x^2y^2+23xy-420$.

Find two numbers whose product is 420, and whose *difference* is 23. These are 35 and 12; hence inserting the signs so that the positive may predominate, we have

$$x^2y^2+23xy-420=(xy+35)(xy-12).$$

EXAMPLES XVII. d.

Resolve into factors:

1. x^2+x-2.
2. x^2-x-6.
3. x^2-x-20.
4. $y^2+4y-12$.
5. $y^2+4y-21$.
6. $y^2-5y-36$.
7. $a^2+8a-33$.
8. $a^2-13a-30$.
9. $a^2+a-132$.
10. $b^2-12b-45$.
11. $b^2+14b-51$.
12. $b^2+10b-39$.
13. m^2-m-56.
14. $m^2-5m-84$.
15. m^2+m-56.
16. $p^2-8p-65$.
17. $p^2+3p-108$.
18. $p^2+p-110$.
19. $x^2+2x-48$.
20. $x^2-7x-120$.
21. $x^2-x-132$.
22. y^4+13y^2-48.
23. $y^2+4xy-96x^2$.
24. $y^2+7xy-98x^2$.
25. $a^4+a^2b^2-72b^4$.
26. $a^2+ab-240b^2$.
27. $14-5a-a^2$.
28. $35-2b-b^2$.
29. $96-4b-b^2$.
30. $72+b-b^2$.

133. We proceed now to the resolution into factors of trinomial expressions when the coefficient of the highest power is not unity.

Again, referring to Chap. v. Art. 48, we may write down the following results:

$$(3x+2)(x+4)=3x^2+14x+8 \quad\ldots\ldots(1),$$
$$(3x-2)(x-4)=3x^2-14x+8 \quad\ldots\ldots(2),$$
$$(3x+2)(x-4)=3x^2-10x-8 \quad\ldots\ldots(3),$$
$$(3x-2)(x+4)=3x^2+10x-8 \quad\ldots\ldots(4).$$

The converse problem presents more difficulty than the cases we have yet considered.

Before endeavouring to give a general method of procedure it will be worth while to examine in detail two of the identities given above.

Consider the result $3x^2-14x+8=(3x-2)(x-4)$.

The first term $3x^2$ is the product of $3x$ and x.

The third term $+8$.................... -2 and -4.

The middle term $-14x$ is the result of adding together the two products $3x\times -4$ and $x\times -2$.

Again, consider the result $3x^2-10x-8=(3x+2)(x-4)$.

The first term $3x^2$ is the product of $3x$ and x.

The third term -8.................... $+2$ and -4.

The middle term $-10x$ is the result of adding together the two products $3x\times -4$ and $x\times 2$; and its sign is negative because the greater of these two products is negative.

134. The beginner will frequently find that it is not easy to select the proper factors at the first trial. Practice alone will enable him to detect at a glance whether any pair he has chosen will combine so as to give the correct coefficients of the expression to be resolved.

Example. Resolve into factors $7x^2 - 19x - 6$.

Write down $(7x \quad 3)(x \quad 2)$ for a first trial, noticing that 3 and 2 must have opposite signs. These factors give $7x^2$ and -6 for the first and third terms. But since $7 \times 2 - 3 \times 1 = 11$, the combination fails to give the correct coefficient of the middle term.

Next try $(7x \quad 2)(x \quad 3)$.

Since $7 \times 3 - 2 \times 1 = 19$, these factors will be correct if we insert the signs so that the negative shall predominate.

Thus $$7x^2 - 19x - 6 = (7x + 2)(x - 3).$$

[Verify by mental multiplication.]

135. In actual work it will not be necessary to put down all these steps at length. The student will soon find that the different cases may be rapidly reviewed, and the unsuitable combinations rejected at once.

It is especially important to pay attention to the two following hints :

1. If the third term of the trinomial is positive, then the second terms of its factors have both the same sign, and this sign is the same as that of the middle term of the trinomial.

2. If the third term of the trinomial is negative, then the second terms of its factors have opposite signs.

Example 1. Resolve into factors $14x^2 + 29x - 15$ ....(1),

$$14x^2 - 29x - 15 \text{ .... } (2).$$

In each case we may write down $(7x \quad 3)(2x \quad 5)$ as a first trial, noticing that 3 and 5 must have opposite signs.

And since $7 \times 5 - 3 \times 2 = 29$, we have only now to insert the proper signs in each factor.

In (1) the positive sign must predominate,

in (2) the negative.............................

Therefore $$14x^2 + 29x - 15 = (7x - 3)(2x + 5).$$

$$14x^2 - 29x - 15 = (7x + 3)(2x - 5).$$

Example 2. Resolve into factors $5x^2+17x+6$ ....(1),
$5x^2-17x+6$ ....(2).

In (1) we notice that the factors which give 6 are both positive.
In (2)...negative.

And therefore for (1) we may write $(5x+\quad)(x+\quad)$.
(2)....................$(5x-\quad)(x-\quad)$.

And, since $5\times3+1\times2=17$, we see that

$$5x^2+17x+6=(5x+2)(x+3).$$
$$5x^2-17x+6=(5x-2)(x-3).$$

Note. In each expression the third term 6 also admits of factors 6 and 1; but this is one of the cases referred to above which the student would reject at once as unsuitable.

EXAMPLES XVII. e.

Resolve into factors:

1. $2a^2+3a+1$.	2. $3a^2+4a+1$.	3. $4a^2+5a+1$.
4. $2a^2+5a+2$.	5. $3a^2+10a+3$.	6. $2a^2+7a+3$.
7. $5a^2+7a+2$.	8. $2a^2+9a+10$.	9. $2a^2+7a+6$.
10. $2x^2+9x+4$.	11. $2x^2+5x-3$.	12. $3x^2+5x-2$.
13. $3y^2+y-2$.	14. $3y^2-7y-6$.	15. $2y^2+9y-5$.
16. $2b^2-5b-3$.	17. $6b^2+7b-3$.	18. $2b^2+b-15$.
19. $4m^2+5m-6$.	20. $4m^2-4m-3$.	21. $6m^2-7m-3$.
22. $4x^2-8xy-5y^2$.	23. $6x^2-7xy+2y^2$.	24. $6x^2-13xy+2y^2$.
25. $12a^2-17ab+6b^2$.	26. $6a^2-5ab-6b^2$.	27. $6a^2+35ab-6b^2$.
28. $2-3y-2y^2$.	29. $3+23y-8y^2$.	30. $8+18y-5y^2$.
31. $4+17x-15x^2$.	32. $6-13a+6a^2$.	33. $28-31b-5b^2$.

When an Expression is the Difference of Two Squares.

136. By multiplying $a+b$ by $a-b$ we obtain the identity

$$(a+b)(a-b)=a^2-b^2,$$

a result which may be verbally expressed as follows:

The product of the sum and the difference of any two quantities is equal to the difference of their squares.

Conversely, *the difference of the squares of any two quantities is equal to the product of the sum and the difference of the two quantities.*

Thus any expression which is the difference of two squares may at once be resolved into factors.

Example. Resolve into factors $25x^2-16y^2$.

$$25x^2-16y^2=(5x)^2-(4y)^2.$$

Therefore the first factor is the sum of $5x$ and $4y$, and the second factor is the difference of $5x$ and $4y$.

$$\therefore \quad 25x^2-16y^2=(5x+4y)(5x-4y).$$

The intermediate steps may usually be omitted.

Example. $\qquad 1-49c^6=(1+7c^3)(1-7c^3).$

The difference of the squares of two numerical quantities is sometimes conveniently found by the aid of the formula

$$a^2-b^2=(a+b)(a-b).$$

Example.
$$\begin{aligned}(329)^2-(171)^2&=(329+171)(329-171)\\&=500\times 158\\&=79000.\end{aligned}$$

EXAMPLES XVII. f.

Resolve into factors:

1. a^2-9. 2. a^2-49. 3. a^2-81. 4. a^2-100.
5. x^2-25. 6. x^2-144. 7. $64-x^2$. 8. $81-4x^2$.
9. $4y^2-1$. 10. y^2-9a^2. 11. $4y^2-25$. 12. $9y^2-49x^2$.
13. $4m^2-81$. 14. $36a^2-1$. 15. k^2-64l^2.
16. $9a^2-25b^2$. 17. $121-16y^2$. 18. $121-36x^2$.
19. $25-c^4$. 20. $a^2b^2-x^2y^2$. 21. $49a^4-100b^2$.
22. $64x^2-49z^2$. 23. $4p^2q^2-81$. 24. $a^4b^4c^2-9$.
25. x^6-4a^4. 26. x^4-25z^4. 27. $a^{10}-p^2q^4$.
28. $16a^{16}-9b^6$. 29. $25x^{12}-4$. 30. $a^6b^8c^4-9x^2$.

Find by factors the value of

31. $(39)^2-(31)^2$. 32. $(51)^2-(49)^2$. 33. $(1001)^2-1$.
34. $(82)^2-(18)^2$. 35. $(275)^2-(225)^2$. 36. $(936)^2-(64)^2$.

When an Expression is the Sum or Difference of Two Cubes.

137. If we divide a^3+b^3 by $a+b$ the quotient is a^2-ab+b^2; and if we divide a^3-b^3 by $a-b$ the quotient is a^2+ab+b^2.

We have therefore the following identities:

$$a^3+b^3=(a+b)(a^2-ab+b^2);$$
$$a^3-b^3=(a-b)(a^2+ab+b^2).$$

These results enable us to resolve into factors any expression which can be written as the sum or the difference of two cubes.

Example 1. $$8x^3 - 27y^3 = (2x)^3 - (3y)^3$$
$$= (2x-3y)(4x^2+6xy+9y^2).$$

Note. The middle term $6xy$ is the *product* of $2x$ and $3y$.

Example 2. $$64a^3+1 = (4a)^3+(1)^3$$
$$= (4a+1)(16a^2-4a+1).$$

We may usually omit the intermediate step and write down the factors at once.

Examples. $343a^6 - 27x^3 = (7a^2-3x)(49a^4+21a^2x+9x^2).$
$8x^9+729 = (2x^3+9)(4x^6-18x^3+81).$

EXAMPLES XVII. g.

Resolve into factors :

1. a^3-b^3.
2. a^3+b^3.
3. $1+x^3$.
4. $1-y^3$.
5. $8x^3+1$.
6. x^3-8z^3.
7. a^3+27b^3.
8. x^3y^3-1.
9. $1-8a^3$.
10. b^3-8.
11. $27+x^3$.
12. $64-p^3$.
13. $125a^3+1$.
14. $216-b^3$.
15. x^3y^3+343.
16. $1000x^3+1$.
17. $512a^3-1$.
18. $a^3b^3c^3-27$.
19. $8x^3-343$.
20. x^3+216y^3.
21. x^6-27z^3.
22. $m^3-1000n^6$.
23. a^3-729b^3.
24. $125a^6+512b^3$.

138. We shall now give some harder applications of the foregoing rules, followed by a miscellaneous exercise in which all the processes of this chapter will be illustrated.

Example 1. Resolve into factors $(a+2b)^2-16x^2$.

The sum of $a+2b$ and $4x$ is $a+2b+4x$,
and their difference is $a+2b-4x$.

$$\therefore \quad (a+2b)^2-16x^2 = (a+2b+4x)(a+2b-4x).$$

If the factors contain like terms they should be collected so as to give the result in its simplest form.

Example 2. $$(3x+7y)^2-(2x-3y)^2$$
$$= \{(3x+7y)+(2x-3y)\}\{(3x+7y)-(2x-3y)\}$$
$$= (3x+7y+2x-3y)(3x+7y-2x+3y)$$
$$= (5x+4y)(x+10y).$$

139. By suitably grouping together the terms, compound expressions can often be expressed as the difference of two squares, and so be resolved into factors.

Example 1. Resolve into factors $9a^2-c^2+4cx-4x^2$.

$$\begin{aligned} 9a^2-c^2+4cx-4x^2 &= 9a^2-(c^2-4cx+4x^2)\\ &= (3a)^2-(c-2x)^2\\ &= (3a+c-2x)(3a-c+2x). \end{aligned}$$

Example 2. Resolve into factors $2bd-a^2-c^2+b^2+d^2+2ac$.

Here the terms $2bd$ and $2ac$ suggest the proper preliminary arrangement of the expression. Thus

$$\begin{aligned} 2bd-a^2-c^2+b^2+d^2+2ac &= b^2+2bd+d^2-a^2+2ac-c^2\\ &= b^2+2bd+d^2-(a^2-2ac+c^2)\\ &= (b+d)^2-(a-c)^2\\ &= (b+d+a-c)(b+d-a+c). \end{aligned}$$

140. The following case is important.

Example. Resolve into factors $x^4+x^2y^2+y^4$.

$$\begin{aligned} x^4+x^2y^2+y^4 &= (x^4+2x^2y^2+y^4)-x^2y^2\\ &= (x^2+y^2)^2-(xy)^2\\ &= (x^2+y^2+xy)(x^2+y^2-xy)\\ &= (x^2+xy+y^2)(x^2-xy+y^2). \end{aligned}$$

141. Sometimes an expression may be resolved into more than two factors.

Example 1. Resolve into factors $16a^4-81b^4$.

$$\begin{aligned} 16a^4-81b^4 &= (4a^2+9b^2)(4a^2-9b^2)\\ &= (4a^2+9b^2)(2a+3b)(2a-3b). \end{aligned}$$

Example 2. Resolve into factors x^6-y^6.

$$\begin{aligned} x^6-y^6 &= (x^3+y^3)(x^3-y^3)\\ &= (x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2). \end{aligned}$$

Note. When an expression can be arranged either as the difference of two squares, or as the difference of two cubes, each of the methods explained in Arts. 136, 137 will be applicable. It will, however, be found simplest to first use the rule for resolving into factors the difference of two squares.

142. In all cases where an expression to be resolved contains a simple factor common to each of its terms, this should be first taken outside a bracket as explained in Art. 129.

Example. Resolve into factors $28x^4y + 64x^3y - 60x^2y$.

$$28x^4y + 64x^3y - 60x^2y = 4x^2y(7x^2 + 16x - 15)$$
$$= 4x^2y(7x - 5)(x + 3).$$

EXAMPLES XVII. h.

Resolve into two or more factors :

1. $(x+y)^2 - z^2$. 2. $(x-y)^2 - z^2$. 3. $(a+2b)^2 - c^2$.
4. $(a+3c)^2 - 1$. 5. $(2x-1)^2 - a^2$. 6. $a^2 - (b+c)^2$.
7. $4a^2 - (b-1)^2$. 8. $9 - (a+x)^2$. 9. $(2a-3b)^2 - c^2$.
10. $(18x+y)^2 - (17x-y)^2$. 11. $(6a+3)^2 - (5a-4)^2$.
12. $4a^2 - (2a-3b)^2$. 13. $x^2 - (2b-3c)^2$.
14. $(x+y)^2 - (m-n)^2$. 15. $(3x+2y)^2 - (2x-3y)^2$.
16. $a^2 - 2ax + x^2 - 4b^2$. 17. $x^2 + a^2 + 2ax - z^2$.
18. $1 - a^2 - 2ab - b^2$. 19. $12xy + 25 - 4x^2 - 9y^2$.
20. $c^2 - a^2 - b^2 + 2ab$. 21. $x^2 - 2x + 1 - m^2 - 4mn - 4n^2$
22. $x^4 + y^4 - z^4 - a^4 + 2x^2y^2 - 2a^2z^2$. 23. $(m+n+p)^2 - (m-n+p)^2$.
24. $a^4 + a^2 + 1$. 25. $a^4b^4 - 16$. 26. $256x^4 - 81y^4$.
27. $16a^4b^2 - b^6$. 28. $64m^7 - mn^6$. 29. $x^4 - x^4y^4$.
30. $a^2b^5 - 81a^2b$. 31. $400a^2x - x^3$. 32. $1 - 729y^6$.
33. $216b^6 + a^3b^3$. 34. $250z^3 + 2$. 35. $1029 - 3x^3$.
36. $ax^3 - ax^2 - 240ax$. 37. $acx^2 + bcx - adx - bd$.
38. $m^4 + 4m^2n^2p^2 + 4n^4p^4$. 39. $8x^2y^3 - x^5$.
40. $6x^3y^2 + 15x^2y^2 - 36xy^2$. 41. $2m^8n^4 - 7m^4n^6 - 4n^8$.
42. $98x^4 - 7x^2y^2 - y^4$. 43. $a^2b^2 - a^2 - b^2 + 1$.
44. $x^3 - 2x^2 - x + 2$. 45. $(a+b)^3 + 1$.
46. $a^2x^3 - 8a^2y^3 - 4b^2x^3 + 32b^2y^3$.
47. $2p - 3q + 4p^2 - 9q^2$. 48. $119 + 10m - m^2$.
49. $24a^2b^2 - 30ab^3 - 36b^4$. 50. $240x^2 + x^6y^4 - x^{10}y^8$.
51. $x^4 + 4x^2 + 16$. 52. $x^4 + y^4 - 7x^2y^2$.
53. $a^4 - 18a^2b^2 + b^4$. 54. $x^8 + x^4 + 1$.

[For additional examples see *Elementary Algebra.*]

Converse Use of Factors.

143. The actual processes of multiplication and division can often be partially or wholly avoided by a skilful use of factors.

It should be observed that the formulæ which the student has seen exemplified in this chapter are just as useful in their converse as in their direct application. Thus the formula for resolving into factors the difference of two squares is equally useful as enabling us to write down at once the product of the sum and the difference of two quantities.

Example 1. Multiply $2a+3b-c$ by $2a-3b+c$.

These expressions may be arranged thus:

$$2a+(3b-c) \text{ and } 2a-(3b-c).$$

Hence the product $= \{2a+(3b-c)\}\{2a-(3b-c)\}$

$$= (2a)^2-(3b-c)^2$$
$$= 4a^2-(9b^2-6bc+c^2)$$
$$= 4a^2-9b^2+6bc-c^2.$$

Example 2. Find the product of

$$x+2,\ x-2,\ x^2-2x+4,\ x^2+2x+4.$$

Taking the first factor with the third, and the second with the fourth,

the product $= \{(x+2)(x^2-2x+4)\}\{(x-2)(x^2+2x+4)\}$

$$= (x^3+8)(x^3-8)$$
$$= x^6-64.$$

Example 3. Divide the product of $2x^2+x-6$ and $6x^2-5x+1$ by $3x^2+5x-2$.

Denoting the division by means of a fraction,

the required quotient $= \dfrac{(2x^2+x-6)(6x^2-5x+1)}{3x^2+5x-2}$

$$= \frac{(2x-3)(x+2)(3x-1)(2x-1)}{(3x-1)(x+2)}$$
$$= (2x-3)(2x-1),$$

by cancelling factors which are common to numerator and denominator.

Example 4. Prove the identity

$$17(5x+3a)^2 - 2(40x+27a)(5x+3a) = 25x^2 - 9a^2.$$

Since each term of the first expression contains the factor $5x+3a$,

$$\begin{aligned} \text{the first side} &= (5x+3a)\{17(5x+3a) - 2(40x+27a)\} \\ &= (5x+3a)(85x+51a-80x-54a) \\ &= (5x+3a)(5x-3a) \\ &= 25x^2 - 9a^2. \end{aligned}$$

EXAMPLES XVII. k.

Employ factors to obtain the product of

1. $a-b+c,\ a-b-c.$
2. $2x-y+z,\ 2x+y+z.$
3. $1+2x-x^2,\ 1-2x-x^2.$
4. $c^2+3c+2,\ c^2-3c-2.$
5. $a+b-c+d,\ a+b+c-d.$
6. $p-q+x-y,\ p-q-x+y.$
7. $a^3-4a^2b+8ab^2-8b^3,\ a^3+4a^2b+8ab^2+8b^3.$

Find the continued product of

8. $(a-b)^2,\ (a+b)^2,\ (a^2+b^2)^2.$
9. $(1-x)^3,\ (1+x)^3,\ (1+x^2)^3.$
10. $a^2-4a+3,\ a^2-a-2,\ a^2+5a+6.$
11. $3-y,\ 3+y,\ 9-3y+y^2,\ 9+3y+y^2.$
12. $1+c+c^2,\ 1-c+c^2,\ 1-c^2+c^4.$
13. Divide $a^3(a+2)(a^2-a-56)$ by a^2+7a.
14. Divide the product of x^2+x-2 and x^2+4x+3 by x^2+5x+6.
15. Divide $3x^2(x+4)(x^2-9)$ by x^2+x-12.
16. Divide the product of $2x^2+11a-21$ and $3a^2-20a-7$ by a^2-49.
17. Divide $(2a^2-a-3)(3a^2-a-2)$ by $6a^2-5a-6$.
18. Divide x^6-7x^3-8 by $(x+1)(x^2+2x+4)$.

Prove the following identities:

19. $(a+b)^3-(a-b)^2(a+b) = 4ab(a+b).$
20. $c^4-d^4-(c-d)^3(c+d) = 2cd(c^2-d^2).$
21. $(m-n)(m+n)^3-m^4+n^4 = 2mn(m^2-n^2).$
22. $(x+y)^4-3xy(x+y)^2 = (x+y)(x^3+y^3).$
23. $3ab(a-b)^2+(a-b)^4 = (a-b)(a^3-b^3).$

MISCELLANEOUS EXAMPLES III.

1. Find the product of $10x^2 - 12 - 3x$ and $2x - 4 + 3x^2$.

2. If $a = 1$, $b = -1$, $c = 2$, $d = 0$, find the value of
$$\frac{a^2 - b^2}{a^2 + b^2} + \frac{b^2 - cd}{2b^2 + cd} + \frac{c^3 - b^3}{3abc}.$$

3. Simplify $2[4x - \{2y + (2x - y) - (x + y)\}]$.

4. Solve the equations:

(1) $\dfrac{x-3}{5} - \dfrac{2-x}{3} = \dfrac{1-2x}{15}$;   (2) $3x - 4y = 25$, $5x + 2y = 7$.

5. Write down the square of $2x^3 - x + 5$.

6. Find the H.C.F. and L.C.M. of $3a^2b^3c$, $12a^4b^2c^3$, $15a^3b^5c$.

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7. Divide $a^4 + 4b^4$ by $a^2 - 2ab + 2b^2$.

8. Find in dollars the price of $5k$ articles at $8a$ cents each.

9. Find the square root of $x^4 - 8x^3 + 24x^2 - 32x + 16$.

10. If $a = 5$, $b = 3$, $c = 1$, find the value of
$$\frac{(a-b)^2}{a+b} + \frac{(b-c)^2}{b+c} + \frac{(a-c)^2}{a+c}.$$

11. Solve $\frac{5}{3}(7x + 5) - 7\frac{2}{3} = 13 - \frac{4}{3}\left(x - \frac{1}{2}\right)$.

12. A is twice as old as B; twenty years ago he was three times as old. Find their ages.

---

13. Simplify $(1 - 2x) - \{3 - (4 - 5x)\} + \{6 - (7 - 8x)\}$.

14. The product of two expressions is
$$6x^4 + 5x^3y + 6x^2y^2 + 5xy^3 + 6y^4,$$
and one of them is $2x^2 + 3xy + 2y^2$; find the other.

15. How old is a boy who $2x$ years ago was half as old as his father now aged 40?

16. Find the lowest common multiple of $2a^2$, $3ab$, $5a^3bc$, $6ab^2c$, $7a^2b$.

17. Find the factors of

(1) $x^2 - xy - 72y^2$.   (2) $6x^2 - 13x + 6$.

18. Find two numbers which differ by 11, and such that one-third of the greater exceeds one-fourth of the less by 7.

---

19. If $a=1$, $b=-1$, $c=2$, $d=0$, find the value of

$$\frac{a+b}{a-b}+\frac{c+d}{c-d}+\frac{ad-bc}{bd+ac}-\frac{c^2-d^2}{a^2+b^2}.$$

20. Simplify $\frac{3}{2}x-y-\left\{2x-\frac{1}{2}y-7-\left(\frac{x}{2}-4\right)+\left(2-\frac{1}{2}x\right)\right\}.$

21. Solve the equations:

(1) $(3x-8)(3x+2)-(4x-11)(2x+1)=(x-3)(x+7)$;

(2) $\frac{x-y}{2}+\frac{x+y}{3}=\frac{25}{6}$, $x+y-5=\frac{2}{3}(y-x)$.

22. A train which travels at the rate of p miles an hour takes q hours between two stations; what will be the rate of a train which takes r hours?

23. Find the sum of

$$\frac{3}{4}a-\frac{1}{3}x,\ 1-\frac{a}{2},\ \frac{2}{3}x-\left(2a-\frac{1}{2}\right),\ \frac{1}{3}x-\frac{1}{4}a.$$

24. Resolve into factors

(1) $12x^2+ax-20a^2$; (2) $a^2-16-6ax+9x^2$.

---

25. Solve

(1) $x+1+2(x+3)=4(x+5)$;

(2) $4x+9y=12$, $6x-3y=7$.

26. Find the value of $\frac{-x+\sqrt{3-2x^2}}{x(1+3x)-x^3}$, when $x=-\frac{1}{3}$.

27. Find the quotient when the product of b^3+c^3 and b^3-c^3 is divided by $b^3-2b^2c+2bc^2-c^3$.

28. A, B, and C have \$168 between them; A's share is greater than B's by \$8, and C's share is three-fourths of A's. Find the share of each.

29. Find the square root of $9x^6-12x^5+22x^4+x^2+12x+4$.

30. Simplify by removing brackets $a^2-[(b-c)^2-\{c^2-(a-b)^2\}]$.

CHAPTER XVIII.

HIGHEST COMMON FACTOR.

144. DEFINITION. The **highest common factor** of two or more algebraical expressions is the *expression of highest dimensions* which divides each of them without remainder.

Note. The term *greatest common measure* is sometimes used instead of *highest common factor*; but this usage is incorrect, for in Algebra our object is to find the factor of *highest dimensions* which is common to two or more expressions, and we are not concerned with the *numerical* values of the expressions or their divisors. The term *greatest common measure* ought to be confined solely to arithmetical quantities, for it can easily be shown by trial that the algebraical highest common factor is not always the greatest common measure.

145. We have already explained how to write down by inspection the highest common factor of two or more *simple* expressions. [See Chap. XII.] An analogous method will enable us readily to find the highest common factor of *compound* expressions which are given as the product of factors, or which can be easily resolved into factors.

Example 1. Find the highest common factor of

$$4cx^3 \text{ and } 2cx^3+4c^2x^2.$$

It will be easy to pick out the common factors if the expressions are arranged as follows:

$$4cx^3 = 4cx^3,$$
$$2cx^3+4c^2x^2 = 2cx^2(x+2c);$$

therefore the H.C.F. is $2cx^2$.

Example 2. Find the highest common factor of

$$3a^2+9ab,\ a^3-9ab^2,\ a^3+6a^2b+9ab^2.$$

Resolving each expression into its factors, we have

$$3a^2+9ab = 3a(a+3b),$$
$$a^3-9ab^2 = a(a+3b)(a-3b),$$
$$a^3+6a^2b+9ab^2 = a(a+3b)(a+3b);$$

therefore the H.C.F. is $a(a+3b)$.

146. When there are two or more expressions containing different powers of the same *compound* factor, the student should be careful to notice that the highest common factor must contain the highest power of the compound factor which is common to all the given expressions.

Example 1. The highest common factor of

$$x(a-x)^2,\ a(a-x)^3, \text{ and } 2ax(a-x)^5 \text{ is } (a-x)^2.$$

Example 2. Find the highest common factor of

$$ax^2+2a^2x+a^3,\ 2ax^2-4a^2x-6a^3,\ 3(ax+a^2)^2.$$

Resolving the expressions into factors, we have

$$ax^2+2a^2x+a^3 = a(x^2+2ax+a^2)$$
$$= a(x+a)^2 \quad\ldots\ldots\ldots\ldots (1),$$
$$2ax^2-4a^2x-6a^3 = 2a(x^2-2ax-3a^2)$$
$$= 2a(x+a)(x-3a) \quad\ldots\ldots\ldots\ldots (2),$$
$$3(ax+a^2)^2 = 3\{a(x+a)\}^2 = 3a^2(x+a)^2 \quad\ldots\ldots\ldots\ldots (3).$$

Therefore from (1), (2), (3), by inspection, the highest common factor is $a(x+a)$.

EXAMPLES XVIII. a.

Find the highest common factor of

1. $x^2-y^2,\ x^2-xy.$
2. $3(a-b)^3,\ a^2-2ab+b^2.$
3. $3a^3-2a^2b,\ 3a^2-2ab.$
4. $9a^2-4b^2,\ 6a^2+4ab.$
5. $c^4-cd^3,\ c^4-c^2d^2.$
6. $x^6-x^4y^2,\ x^3y^2+x^2y^3.$
7. $a^2x^3(a-x)^3,\ 2a^2x^2(a-x)^2.$
8. $2x^2-8x+8,\ (x-2)^3.$
9. $ax+x,\ a^4x+ax.$
10. $x^2y^2-y^4,\ xy^2+y^3,\ xy-y^2.$
11. $x^3+xy^2,\ x^2+xy,\ x^2y+xy^2.$
12. $x^3y^3-y^6,\ y^2(xy-y^2)^2.$
13. $(a^2-ax)^2,\ (ax-x^2)^3.$
14. $(abc-bc^2)^2,\ (a^2c-ac^2)^2.$
15. $x^3-x^2-42x,\ x^4-49x^2.$
16. $(x^3-5x^2)^2,\ x^5-8x^4+15x^3.$
17. $a^3-36a,\ a^3+2a^2-48a.$
18. $3a^2+7a-6,\ 2a^2+7a+3.$
19. $2x^2-9x+4,\ 3x^2-7x-20.$
20. $3c^4+5c^3-12c^2,\ 6c^5+7c^4-20c^3.$
21. $4m^4-9m^2,\ 6m^3-5m^2-6m,\ 6m^4+5m^3-6m^2.$
22. $3a^4x^3-8a^3x^3+4a^2x^3,\ 3a^5x^2-11a^4x^2+6a^3x^2,$
$3a^4x^3+16a^3x^3-12a^2x^3.$

147. The highest common factor should always be determined by inspection when possible, but it will sometimes happen that expressions cannot be readily resolved into factors. To find the highest common factor in such cases, we adopt a method analogous to that used in Arithmetic for finding the greatest common measure of two or more numbers.

148. We shall now work out examples illustrative of the algebraical process of finding the highest common factor; for the proof of the rules the reader may consult the *Elementary Algebra*, Arts. 102, 103. We may here conveniently *enunciate* two principles, which the student should bear in mind in reading the examples which follow.

I. *If an expression contain a certain factor, any multiple of the expression is divisible by that factor.*

II. *If two expressions have a common factor, it will divide their sum and their difference; and also the sum and the difference of any multiples of them.*

Example. Find the highest common factor of

$$4x^3-3x^2-24x-9 \text{ and } 8x^3-2x^2-53x-39.$$

$$\begin{array}{r|r|r|l}
x & 4x^3-3x^2-24x-9 & 8x^3-2x^2-53x-39 & 2\\
 & 4x^3-5x^2-21x & 8x^3-6x^2-48x-18 & \\
\hline
2x & 2x^2-3x-9 & 4x^2-5x-21 & 2\\
 & 2x^2-6x & 4x^2-6x-18 & \\
\hline
3 & 3x-9 & x-3 & \\
 & 3x-9 & & \\
\end{array}$$

Therefore the H.C.F. is $x-3$.

Explanation. First arrange the given expressions according to descending or ascending powers of x. The expressions so arranged having their first terms of the same order, we take for divisor that whose highest power has the smaller coefficient. Arrange the work in parallel columns as above. When the first remainder $4x^2-5x-21$ is made the divisor we put the quotient x to the *left* of the dividend. Again, when the second remainder $2x^2-3x-9$ is in turn made the divisor, the quotient 2 is placed to the *right*; and so on. As in Arithmetic, the last divisor $x-3$ is the highest common factor required.

149. This method is only useful to determine the *compound* factor of the highest common factor. Simple factors of the given expressions must be first removed from them, and the highest common factor of these, if any, must be observed and multiplied into the *compound* factor given by the rule.

Example. Find the highest common factor of

$$24x^4 - 2x^3 - 60x^2 - 32x \text{ and } 18x^4 - 6x^3 - 39x^2 - 18x.$$

We have $24x^4 - 2x^3 - 60x^2 - 32x = 2x(12x^3 - x^2 - 30x - 16)$,

and $\qquad 18x^4 - 6x^3 - 39x^2 - 18x = 3x(6x^3 - 2x^2 - 13x - 6)$.

Also $2x$ and $3x$ have the common factor x. Removing the simple factors $2x$ and $3x$, and *reserving* their common factor x, we continue as in Art. 148.

$$\begin{array}{r|r|r|r}
2x & 6x^3 - 2x^2 - 13x - 6 & 12x^3 - x^2 - 30x - 16 & 2 \\
 & 6x^3 - 8x^2 - 8x & 12x^3 - 4x^2 - 26x - 12 & \\
\hline
2 & 6x^2 - 5x - 6 & 3x^2 - 4x - 4 & x \\
 & 6x^2 - 8x - 8 & 3x^2 + 2x & \\
\hline
 & 3x + 2 & -6x - 4 & -2 \\
 & & -6x - 4 &
\end{array}$$

Therefore the H. C. F. is $x(3x+2)$.

150. So far the process of Arithmetic has been found exactly applicable to the algebraical expressions we have considered. But in many cases certain modifications of the arithmetical method will be found necessary. These will be more clearly understood if it is remembered that, at every stage of the work, the remainder must contain as a factor of itself the highest common factor we are seeking. [See Art. 148, I. & II.]

Example 1. Find the highest common factor of

$$3x^3 - 13x^2 + 23x - 21 \text{ and } 6x^3 + x^2 - 44x + 21.$$

$$\begin{array}{r|r|l}
3x^3 - 13x^2 + 23x - 21 & 6x^3 + x^2 - 44x + 21 & 2 \\
 & 6x^3 - 26x^2 + 46x - 42 & \\
\hline
 & 27x^2 - 90x + 63 &
\end{array}$$

Here on making $27x^2 - 90x + 63$ a divisor, we find that it is not contained in $3x^3 - 13x^2 + 23x - 21$ with an *integral* quotient. But noticing that $27x^2 - 90x + 63$ may be written in the form $9(3x^2 - 10x + 7)$, and also bearing in mind that every remainder in the course of the work contains the H. C. F., we conclude that the H. C. F. we are seeking is contained in $9(3x^2 - 10x + 7)$. But the two original expressions have no *simple* factors, therefore their H. C. F. can have none. We may therefore *reject* the factor 9 and go on with divisor $3x^2 - 10x + 7$.

Resuming the work, we have

$$\begin{array}{r|r|r|l}
x & 3x^3-13x^2+23x-21 & 3x^2-10x+7 & x \\
 & 3x^3-10x^2+\ \ 7x & 3x^2-\ \ 7x & \\ \hline
-1 & -\ 3x^2+16x-21 & -\ 3x+7 & -1 \\
 & -\ 3x^2+10x-\ \ 7 & -\ 3x+7 & \\ \hline
 & 2\,)\,6x-14 & & \\
 & 3x-\ \ 7 & &
\end{array}$$

Therefore the highest common factor is $3x-7$.

The factor 2 has been removed on the same grounds as the factor 9 above.

151. Sometimes the process is more convenient when the expressions are arranged in ascending powers.

Example. Find the highest common factor of

$$3-4a-16a^2-9a^3 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

and
$$4-7a-19a^2-8a^3 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

As the expressions stand we cannot begin to divide one by the other without using a fractional quotient. The difficulty may be obviated by *introducing* a suitable factor, just as in the last case we found it useful to remove a factor when we could no longer proceed with the division in the ordinary way. The given expressions have no common *simple* factor, hence their H.C.F. cannot be affected if we multiply either of them by any simple factor.

Multiply (1) by 4 and use (2) as a divisor:

$$\begin{array}{r|r|r|l}
 & 4-\ 7a-\ 19a^2-\ \ 8a^3 & 12-16a-64a^2-36a^3 & 3 \\
 & 5 & 12-21a-57a^2-24a^3 & \\ \hline
4 & 20-35a-\ 95a^2-\ 40a^3 & a\,)\,5a-\ 7a^2-12a^3 & \\
 & 20-28a-\ 48a^2 & 5-\ 7a\ -12a^2 & 5 \\ \hline
 & -\ 7a-\ 47a^2-\ 40a^3 & 5+\ 5a & \\
 & -5 & -12a\ -12a^2 & -12a \\ \hline
7a & 35a+235a^2+200a^3 & -12a\ -12a^2 & \\
 & 35a-\ 49a^2-\ 84a^3 & & \\ \hline
 & 284a^2\,)\,284a^2+284a^3 & & \\
 & 1+a & &
\end{array}$$

Therefore the H.C.F. is $1+a$.

After the first division the factor a is removed as explained in Art. 150; then the factor 5 is introduced because the first term of $4-7a-19a^2-8a^3$ is not divisible by the first term of $5-7a-12a^2$. At the next stage a factor -5 is introduced, and finally the factor $284a^2$ is removed.

152. From the last two examples it appears that we may multiply or divide either of the given expressions, or any of the remainders which occur in the course of the work, by any factor which does not divide both of the given expressions.

EXAMPLES XVIII. b.

Find the highest common factor of

1. $2x^3+3x^2+x+6,\ 2x^3+x^2+2x+3.$
2. $2y^3-9y^2+9y-7,\ y^3-5y^2+5y-4.$
3. $2x^3+8x^2-5x-20,\ 6x^3-4x^2-15x+10.$
4. $a^3+3a^2-16a+12,\ a^3+a^2-10a+8.$
5. $6x^3-x^2-7x-2,\ 2x^3-7x^2+x+6.$
6. $q^3-3q+2,\ q^3-5q^2+7q-3.$
7. $a^4+a^3-2a^2+a-3,\ 5a^3+3a^2-17a+6.$
8. $3y^4-3y^3-15y^2-9y,\ 4y^5-16y^4-44y^3-24y^2.$
9. $15x^4-15x^3+10x^2-10x,\ 30x^5+120x^4+20x^3+80x^2.$
10. $2m^4+7m^3+10m^2+35m,\ 4m^4+14m^3-4m^2-6m+28.$
11. $3x^4-9x^3+12x^2-12x,\ 6x^3-6x^2-15x+6.$
12. $2a^5-4a^4-6a,\ a^5+a^4-3a^3-3a^2.$
13. $x^3+4x^2-2x-15,\ x^3-21x-36.$
14. $9a^4+2a^2x^2+x^4,\ 3a^4-8a^3x+5a^2x^2-2ax^3.$
15. $2-3a+5a^2-2a^3,\ 2-5a+8a^2-3a^3.$
16. $3x^2-5x^3-15x^4-4x^5,\ 6x-7x^2-29x^3-12x^4.$

[For additional examples see *Elementary Algebra*.]

CHAPTER XIX.

FRACTIONS.

153. THE principles explained in Chapter XVIII. may now be applied to the reduction and simplification of fractions.

Reduction to Lowest Terms.

154. Rule. *The value of a fraction is not altered if we multiply or divide the numerator and denominator by the same quantity.*

An algebraical fraction may therefore be reduced to an equivalent fraction by dividing numerator and denominator by any common factor; if this factor be the highest common factor, the resulting fraction is said to be in its *lowest terms.*

Example 1. Reduce to lowest terms $\dfrac{24a^3c^2x^2}{18a^3x^2 - 12a^2x^3}$.

$$\text{The expression} = \frac{24a^3c^2x^2}{6a^2x^2(3a - 2x)}$$

$$= \frac{4ac^2}{3a - 2x}.$$

Example 2. Reduce to lowest terms $\dfrac{6x^2 - 8xy}{9xy - 12y^2}$.

$$\text{The expression} = \frac{2x(3x - 4y)}{3y(3x - 4y)} = \frac{2x}{3y}.$$

Note. The beginner should be careful not to begin cancelling until he has expressed both numerator and denominator in the most convenient form, by resolution into factors where necessary.

EXAMPLES XIX. a.

Reduce to lowest terms:

1. $\dfrac{3x^2}{6x^2 - 3xy}$. 2. $\dfrac{a^2 - 2a}{4a^3 - 8a^2}$. 3. $\dfrac{3ab + b^2}{6a^2b^2 + 2ab^3}$.

Reduce to lowest terms:

4. $\dfrac{5x^2yz^2}{5x^2y+10x^2z}$. 5. $\dfrac{9x^2-y^2}{6x^2y-2xy^2}$. 6. $\dfrac{2x^2y^2-8}{3x^2y+6x}$.

7. $\dfrac{x^2+4x}{x^2+x-12}$. 8. $\dfrac{7a^2x-7a^2c}{5cx^2-10c^2x+5c^3}$. 9. $\dfrac{x^2+x-30}{5x^2+30x}$.

10. $\dfrac{(2a+b)^2}{4a^3-ab^2}$. 11. $\dfrac{a^3+b^3}{a^2-ab-2b^2}$. 12. $\dfrac{2c^2+5cd-3d^2}{c^2+6cd+9d^2}$.

13. $\dfrac{x^2-4x-21}{3x^2+10x+3}$. 14. $\dfrac{x^2-2x-15}{3x^2-12x-15}$. 15. $\dfrac{2x^2+x-3}{2x^2+11x+12}$.

16. $\dfrac{3a^3-24}{4a^2+4a-24}$. 17. $\dfrac{4x^3-25xy^2}{2x^2+xy-15y^2}$. 18. $\dfrac{18a^3+6a^2x+2ax^2}{27a^3-x^3}$.

155. When the factors of the numerator and denominator cannot be determined by inspection, the fraction may be reduced to its lowest terms by dividing both numerator and denominator by the highest common factor, which may be found by the rules given in Chap. XVIII.

Example. Reduce to lowest terms $\dfrac{3x^3-13x^2+23x-21}{15x^3-38x^2-2x+21}$.

The H.C.F. of numerator and denominator is $3x-7$.

Dividing numerator and denominator by $3x-7$, we obtain as respective quotients x^2-2x+3 and $5x^2-x-3$.

Thus $\dfrac{3x^3-13x^2+23x-21}{15x^3-38x^2-2x+21}=\dfrac{(3x-7)(x^2-2x+3)}{(3x-7)(5x^2-x-3)}=\dfrac{x^2-2x+3}{5x^2-x-3}$.

156. If either numerator or denominator can readily be resolved into factors we may use the following method.

Example. Reduce to lowest terms $\dfrac{x^3+3x^2-4x}{7x^3-18x^2+6x+5}$.

The numerator $=x(x^2+3x-4)=x(x+4)(x-1)$.

Of these factors the only one which can be a common divisor is $x-1$. Hence, arranging the denominator so as to shew $x-1$ as a factor,

$$\text{the fraction } =\frac{x(x+4)(x-1)}{7x^2(x-1)-11x(x-1)-5(x-1)}$$

$$=\frac{x(x+4)(x-1)}{(x-1)(7x^2-11x-5)}=\frac{x(x+4)}{7x^2-11x-5}.$$

EXAMPLES XIX. b.

Reduce to lowest terms :

1. $\dfrac{x^3-x^2+2x-2}{3x^4+7x^2+2}$.

2. $\dfrac{a^3+a+2}{a^3-4a^2+5a-6}$.

3. $\dfrac{y^3-2y^2-2y-3}{3y^3+4y^2+4y+1}$.

4. $\dfrac{m^3-m^2-2m}{m^3-m^2-m-2}$.

5. $\dfrac{a^3-2ab^2+21b^3}{a^3-4a^2b-21ab^2}$.

6. $\dfrac{9x^3-a^2x-2a^3}{3x^3-10ax^2-7a^2x-4a^3}$.

7. $\dfrac{5x^3-4x-1}{2x^3-3x^2+1}$.

8. $\dfrac{c^3+2c^2d-12cd^2-9d^3}{2c^3+6c^2d-28cd^2-24d^3}$.

9. $\dfrac{x^4-21x+8}{8x^4-21x^3+1}$.

10. $\dfrac{y^5+6y^4+2y^3-9y^2}{y^4+7y^3+3y^2-11y}$.

11. $\dfrac{1-x^2+6x^3}{2-x+9x^3}$.

12. $\dfrac{2-5x-4x^2+3x^3}{4+4x+9x^2+4x^3-5x^4}$.

[For additional examples see *Elementary Algebra.*]

Multiplication and Division of Fractions.

157. **Rule.** *To multiply together two or more fractions: multiply the numerators for a new numerator, and the denominators for a new denominator.*

Thus $$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}.$$

Similarly, $$\frac{a}{b}\times\frac{c}{d}\times\frac{e}{f}=\frac{ace}{bdf};$$

and so for any number of fractions.

In practice the application of this rule is modified by removing in the course of the work factors which are common to numerator and denominator.

Example. Simplify $\dfrac{2a^2+3a}{4a^3}\times\dfrac{4a^2-6a}{12a+18}$.

$$\text{The expression}=\frac{a(2a+3)}{4a^3}\times\frac{2a(2a-3)}{6(2a+3)}$$
$$=\frac{2a-3}{12a},$$

by cancelling those factors which are common to both numerator and denominator.

158. **Rule.** *To divide one fraction by another: invert the divisor, and proceed as in multiplication.*

Thus $$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}.$$

Example. Simplify $\dfrac{6x^2 - ax - 2a^2}{ax - a^2} \times \dfrac{x - a}{9x^2 - 4a^2} \div \dfrac{2x + a}{3ax + 2a^2}$.

$$\text{The expression} = \frac{6x^2 - ax - 2a^2}{ax - a^2} \times \frac{x - a}{9x^2 - 4a^2} \times \frac{3ax + 2a^2}{2x + a}$$

$$= \frac{(3x - 2a)(2x + a)}{a(x - a)} \times \frac{x - a}{(3x + 2a)(3x - 2a)} \times \frac{a(3x + 2a)}{2x + a}$$

$$= 1,$$

since all the factors cancel each other.

EXAMPLES XIX. c.

Simplify

1. $\dfrac{x^2 - 1}{x^2 + 3x} \times \dfrac{2x^3 + 6x^2}{x^2 + x}$.

2. $\dfrac{ab + 2}{4a^2 - 12ab} \times \dfrac{a^2b - 3ab^2}{a^2b^2 - 4}$.

3. $\dfrac{2c^2 + 3cd}{4c^2 - 9d^2} \div \dfrac{c + d}{2cd - 3d^2}$.

4. $\dfrac{5y - 10y^2}{12y^2 + 6y^3} \div \dfrac{1 - 2y}{2y + y^2}$.

5. $\dfrac{x^2 - 4}{x^2 + 4x + 4} \div \dfrac{x - 2}{x + 2}$.

6. $\dfrac{b^2 - 5b}{3b - 4a} \times \dfrac{9b^2 - 16a^2}{b^2 - 25}$.

7. $\dfrac{x^2 + 9x + 20}{x^2 + 5x + 4} \div \dfrac{x^2 + 7x + 10}{x^2 + 3x + 2}$.

8. $\dfrac{y^2 - y - 12}{y^2 - 16} \times \dfrac{y^2 - 2y - 24}{y^2 + 6y + 9}$.

9. $\dfrac{a^3 + 27}{a^2 + 9a + 14} \div \dfrac{a^2 - 4a - 21}{a^2 - 49}$.

10. $\dfrac{2a^2 - 3a - 2}{a^2 - a - 6} \times \dfrac{3a^2 - 8a - 3}{3a^2 - 5a - 2}$.

11. $\dfrac{b^3 + 125}{5b^2 + 24b - 5} \times \dfrac{25b^2 - 1}{b^3 - 5b^2 + 25b}$.

12. $\dfrac{3m^2 - m - 2}{3m^2 + 8m + 4} \div \dfrac{4m^2 + m - 5}{m + 2}$.

13. $\dfrac{2p^2 + 4p}{p^2 - 9} \times \dfrac{p^2 - 5p + 6}{p^2 - 5p} \times \dfrac{p^2 - 2p - 15}{p^2 - 4}$.

14. $\dfrac{64a^2b^2 - 1}{x^2 - x - 56} \times \dfrac{x^2 - 49}{8a^3b - a^2} \div \dfrac{x - 7}{a^2x - 8a^2}$.

15. $\dfrac{4x^2 + 4x - 15}{x^2 + 2x - 48} \times \dfrac{x + 8}{2x^2 - 15x + 18} \div \dfrac{2x^2 + 5x}{(x - 6)^2}$.

16. $\dfrac{a^2 + 8ab - 9b^2}{a^2 + 6ab - 27b^2} \times \dfrac{a^2 - 7ab + 12b^2}{a^3 - b^3} \times \dfrac{a^3 + a^2b + ab^2}{a^2 - 3ab - 4b^2}$.

17. $\dfrac{ax^2 - 16a^3}{x^2 - ax - 30a^2} \times \dfrac{x^2 + ax - 20a^2}{ax^2 + 9a^2x + 20a^3} \div \dfrac{x^2 - 8ax + 16a^2}{x^2 + 8ax + 15a^2}$.

18. $\dfrac{(a - b)^2 - c^2}{a^2 - ab + ac} \times \dfrac{a^2 + ab + ac}{(a - c)^2 - b^2} \times \dfrac{(a + b)^2 - c^2}{(a + b + c)^2}$.

[For additional examples see *Elementary Algebra.*]

CHAPTER XX.

Lowest Common Multiple.

159. Definition. The **lowest common multiple** of two or more algebraical expressions is *the expression of lowest dimensions* which is divisible by each of them without remainder.

The lowest common multiple of compound expressions which are given as the product of factors, or which can be easily resolved into factors, can be readily found by inspection.

Example 1. The lowest common multiple of $6x^2(a-x)^2$, $8a^3(a-x)^3$, and $12ax(a-x)^5$ is $24a^3x^2(a-x)^5$.

For it consists of the product of

(1) the L.C.M. of the numerical coefficients;

(2) the lowest power of each factor which is divisible by every power of that factor occurring in the given expressions.

Example 2. Find the lowest common multiple of

$$3a^2+9ab,\ \ 2a^3-18ab^2,\ \ a^3+6a^2b+9ab^2.$$

Resolving each expression into its factors, we have

$$\begin{aligned} 3a^2+9ab &= 3a(a+3b), \\ 2a^3-18ab^2 &= 2a(a+3b)(a-3b), \\ a^3+6a^2b+9ab^2 &= a(a+3b)(a+3b) \\ &= a(a+3b)^2. \end{aligned}$$

Therefore the L.C.M. is $6a(a+3b)^2(a-3b)$.

Example 3. Find the lowest common multiple of

$$(yz^2-xyz)^2,\ \ y^2(xz^2-x^3),\ \ z^4+2xz^3+x^2z^2.$$

Resolving each expression into its factors, we have

$$\begin{aligned} (yz^2-xyz)^2 &= \{yz(z-x)\}^2 = y^2z^2(z-x)^2, \\ y^2(xz^2-x^3) &= y^2x(z^2-x^2) = xy^2(z-x)(z+x), \\ z^4+2xz^3+x^2z^2 &= z^2(z^2+2xz+x^2) = z^2(z+x)^2. \end{aligned}$$

Therefore the L.C.M. is $xy^2z^2(z+x)^2(z-x)^2$.

EXAMPLES XX. a.

Find the lowest common multiple of

1. $a^2,\ a^3-a^2$.
2. $x^2,\ x^2-3x^3$.
3. $4m^2,\ 6m^3-8m^2$.
4. $6x^2,\ x^4+3x^2$.
5. $b^2+b,\ b^3-b$.
6. $x^2-4,\ x^3+8$.
7. $9a^2b-b,\ 6a^2+2a$.
8. $k^2-k+1,\ k^3-1$.
9. $m^2-5m+6,\ m^2+5m-14$.
10. $y^2+3y^3,\ y^3-9y^5$.
11. $x^2-9x+14,\ x^2+4x-12$.
12. $x^3+27y^3,\ x^2+xy-6y^2$.
13. $b^2+9b+20,\ b^2+b-20$.
14. $c^2-3cx-18x^2,\ c^2-8cx+12x^2$.
15. $a^2-4a-5,\ a^2-8a+15,\ a^3-2a^2-3a$.
16. $2x^2-4xy-16y^2,\ x^2-6xy+8y^2,\ 3x^2-12y^2$.
17. $3x^3-12a^2x,\ 4x^2+16ax+16a^2$.
18. $a^5c-a^3c^3,\ (a^2c+ac^2)^2$.
19. $(a^2x-2ax^2)^2,\ (2ax-4x^2)^2$.
20. $(2a-a^2)^3,\ 4a^2-4a^3+a^4$.
21. $2x^2-x-3,\ (2x-3)^2,\ 4x^2-9$.
22. $2x^2-7x-4,\ 6x^2-7x-5,\ x^3-8x^2+16x$.
23. $10x^2y^2(x^3-y^3),\ 15y^4(x-y)^3,\ 12x^3y(x-y)(x^2-y^2)$.
24. $2x^2+x-6,\ 7x^2+11x-6,\ (7x^2-3x)^2$.
25. $6a^3-7a^2x-3ax^2,\ 10a^2x-11ax^2-6x^3,\ 10a^2-21ax-10x^2$.

160. When the given expressions are such that their factors cannot be determined by inspection, they must be resolved by finding the highest common factor.

Example. Find the lowest common multiple of

$$2x^4+x^3-20x^2-7x+24 \text{ and } 2x^4+3x^3-13x^2-7x+15.$$

The highest common factor is x^2+2x-3.

By division, we obtain

$$2x^4+x^3-20x^2-7x+24=(x^2+2x-3)(2x^2-3x-8).$$
$$2x^4+3x^3-13x^2-7x+15=(x^2+2x-3)(2x^2-x-5).$$

Therefore the L.C.M. is $(x^2+2x-3)(2x^2-3x-8)(2x^2-x-5)$.

EXAMPLES XX. b.

Find the lowest common multiple of

1. $x^3 - 2x^2 - 13x - 10$ and $x^3 - x^2 - 10x - 8$.

2. $y^3 + 3y^2 - 3y - 9$ and $y^3 + 3y^2 - 8y - 24$.

3. $m^3 + 3m^2 - m - 3$ and $m^3 + 6m^2 + 11m + 6$.

4. $2x^4 - 2x^3 + x^2 + 3x - 6$ and $4x^4 - 2x^3 + 3x - 9$.

5. Find the highest common factor and the lowest common multiple of $(x - x^2)^3$, $(x^2 - x^3)^2$, $x^3 - x^4$.

6. Find the lowest common multiple of $(a^4 - a^2x^2)^2$, $(a^2 + ax)^3$, $(ax - x^2)^2$.

7. Find the highest common factor and lowest common multiple of $6x^2 + 5x - 6$ and $6x^2 + x - 12$; and show that the product of the H.C.F. and L.C.M. is equal to the product of the two given expressions.

8. Find the highest common factor and the lowest common multiple of $a^2 + 5ab + 6b^2$, $a^2 - 4b^2$, $a^3 - 3ab^2 + 2b^3$.

9. Find the lowest common multiple of $1 - x^2 - x^4 + x^5$ and $1 + 2x + x^2 - x^4 - x^5$.

10. Find the highest common factor of $(a^3 - 4ab^2)^2$, $(a^3 + 2a^2b)^3$, $(a^2x + 2abx)^2$.

11. Find the highest common factor and the lowest common multiple of $(3a^2 - 2ax)^2$, $2a^2x(9a^2 - 4x^2)$, $6a^3x - 13a^2x^2 + 6ax^3$.

12. Find the lowest common multiple of $x^3 + x^2y + xy^2$, $x^3y - y^4$, $x^5y + x^3y^3 + xy^5$.

[For additional examples see *Elementary Algebra.*]

CHAPTER XXI.

ADDITION AND SUBTRACTION OF FRACTIONS.

161. To find the algebraical sum of a number of fractions we must, as in Arithmetic, first reduce them to a common denominator. For this purpose it is usually most convenient to take the *lowest* common denominator.

Rule. *To reduce fractions to their lowest common denominator: find the L.C.M. of the given denominators, and take it for the common denominator; divide it by the denominator of the first fraction, and multiply the numerator of this fraction by the quotient so obtained; and do the same with all the other given fractions.*

Example. Express with lowest common denominator

$$\frac{5x}{2a(x-a)} \text{ and } \frac{4a}{3x(x^2-a^2)}.$$

The lowest common denominator is $6ax(x-a)(x+a)$.

We must therefore multiply the numerators by $3x(x+a)$ and $2a$ respectively.

Hence the equivalent fractions are

$$\frac{15x^2(x+a)}{6ax(x-a)(x+a)} \text{ and } \frac{8a^2}{6ax(x-a)(x+a)}.$$

162. We may now enunciate the rule for the addition or subtraction of fractions.

Rule. *To add or subtract fractions: reduce them to the lowest common denominator; find the algebraical sum of the numerators, and retain the common denominator.*

Thus
$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd},$$

and
$$\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}.$$

163. We begin with examples in further illustration of those already discussed in Chapter XII.

Example 1. Find the value of $\dfrac{2x+a}{3a}+\dfrac{5x^2-4ax}{9a^2}$.

The lowest common denominator is $9a^2$.

Therefore the expression $=\dfrac{3a(2x+a)+5x^2-4ax}{9a^2}$

$$=\frac{6ax+3a^2+5x^2-4ax}{9a^2}=\frac{3a^2+2ax+5x^2}{9a^2}.$$

Example 2. Find the value of $\dfrac{x-2y}{xy}+\dfrac{3y-a}{ay}-\dfrac{3x-2a}{ax}$.

The lowest common denominator is axy.

Thus the expression $=\dfrac{a(x-2y)+x(3y-a)-y(3x-2a)}{axy}$

$$=\frac{ax-2ay+3xy-ax-3xy+2ay}{axy}$$

$$=0,$$

since the terms in the numerator destroy each other.

Note. To ensure accuracy the beginner is recommended to use brackets as in the first line of work above.

EXAMPLES XXI. a.

Find the value of

1. $\dfrac{a-2}{3}+\dfrac{a-1}{2}+\dfrac{a+5}{6}$.

2. $\dfrac{3x-1}{4}+\dfrac{x+3}{6}+\dfrac{2x-1}{3}$.

3. $\dfrac{2b-1}{5}+\dfrac{b-3}{2}-\dfrac{7b+3}{10}$.

4. $\dfrac{2m-5}{9}-\dfrac{m+3}{6}+\dfrac{m-5}{12}$.

5. $-\dfrac{2x-1}{5}+\dfrac{3x-1}{7}-\dfrac{x-2}{35}$.

6. $\dfrac{2x-5}{x}-\dfrac{x-4}{x}-\dfrac{x^2-4x}{3x^2}$.

7. $\dfrac{y-z}{yz}+\dfrac{z-x}{zx}+\dfrac{x-y}{xy}$.

8. $-\dfrac{a+x}{2a}+\dfrac{a+2x}{3a}-\dfrac{x-5a}{6a}$.

9. $\dfrac{2a^2-5a}{a}-\dfrac{a^3+3a^2}{a^2}+\dfrac{9a^3-a^4}{a^3}$.

10. $-\dfrac{x-y}{y}+\dfrac{x+y}{x}-\dfrac{6xy-4x^2}{3xy}$.

11. $\dfrac{a^3-2b^3-c^3}{5b^3}-\dfrac{3a^3-3c^3}{15b^3}$.

12. $\dfrac{ab-bc}{2bc}-\dfrac{a}{3c}-\dfrac{2a^2-ab}{2ab}$.

13. $\dfrac{2ay-xy+4x}{2xy}-1-\dfrac{a}{2x}$.

14. $\dfrac{a^2-ab}{a^2b}-\dfrac{b-c}{bc}-\dfrac{2c^2-ac}{c^2a}$.

164. We shall now consider the addition and subtraction of fractions whose denominators are compound expressions. *The lowest common multiple of the denominators should always be written down by inspection when possible.*

Example 1. Simplify $\dfrac{2x-3a}{x-2a}-\dfrac{2x-a}{x-a}$.

The lowest common denominator is $(x-2a)(x-a)$.

Hence, multiplying the numerators by $x-a$ and $x-2a$ respectively, we have

$$\text{the expression}=\frac{(2x-3a)(x-a)-(2x-a)(x-2a)}{(x-2a)(x-a)}$$

$$=\frac{2x^2-5ax+3a^2-(2x^2-5ax+2a^2)}{(x-2a)(x-a)}$$

$$=\frac{2x^2-5ax+3a^2-2x^2+5ax-2a^2}{(x-2a)(x-a)}$$

$$=\frac{a^2}{(x-2a)(x-a)}.$$

Note. In finding the value of such an expression as

$$-(2x-a)(x-2a),$$

the beginner should first express the product in brackets, and then remove the brackets, as we have done. After a little practice he will be able to take both steps together.

Example 2. Find the value of $\dfrac{3x+2}{x^2-16}+\dfrac{x-5}{(x+4)^2}$.

The lowest common denominator is $(x-4)(x+4)^2$.

$$\text{Hence the expression}=\frac{(3x+2)(x+4)+(x-5)(x-4)}{(x-4)(x+4)^2}$$

$$=\frac{3x^2+14x+8+x^2-9x+20}{(x-4)(x+4)^2}$$

$$=\frac{4x^2+5x+28}{(x-4)(x+4)^2}.$$

165. If a fraction is not in its lowest terms, it should be simplified before it is combined with other fractions.

Example. Simplify $\dfrac{x^2+5xy-4y^2}{x^2-16y^2}-\dfrac{xy-3y^2}{x^2+xy-12y^2}$.

$$\text{The expression} = \frac{x^2+5xy-4y^2}{x^2-16y^2}-\frac{y(x-3y)}{(x+4y)(x-3y)}$$
$$=\frac{x^2+5xy-4y^2}{x^2-16y^2}-\frac{y}{x+4y}$$
$$=\frac{x^2+5xy-4y^2-y(x-4y)}{x^2-16y^2}$$
$$=\frac{x^2+5xy-4y^2-xy+4y^2}{x^2-16y^2}$$
$$=\frac{x^2+4xy}{x^2-16y^2}=\frac{x(x+4y)}{(x+4y)(x-4y)}=\frac{x}{x-4y}.$$

EXAMPLES XXI. b.

Find the value of

1. $\dfrac{1}{a-2}+\dfrac{1}{a-3}$.

2. $\dfrac{1}{x-4}-\dfrac{1}{x-2}$.

3. $\dfrac{1}{b-2}-\dfrac{1}{b+2}$.

4. $\dfrac{a}{x-a}-\dfrac{b}{x-b}$.

5. $\dfrac{a-x}{a+x}+\dfrac{a+x}{a-x}$.

6. $\dfrac{a+3}{a-3}-\dfrac{a-3}{a+3}$.

7. $\dfrac{x}{x-1}-\dfrac{x^2}{x^2-1}$.

8. $\dfrac{3a}{a^2-4}-\dfrac{1}{a+2}$.

9. $\dfrac{x^2}{x^2-4y^2}+\dfrac{x-2y}{x+2y}$.

10. $\dfrac{1}{a(a-b)}+\dfrac{1}{a(a+b)}$.

11. $\dfrac{3a}{2x(x-a)}-\dfrac{2a}{3x(x+a)}$.

12. $\dfrac{5}{x-2}-\dfrac{4x}{(x-2)(x+1)}$.

13. $\dfrac{1}{y^2-2y-3}+\dfrac{3(y+2)}{y^2-y-6}$.

14. $\dfrac{1}{1-a}+\dfrac{a}{(1-a)^2}$.

15. $\dfrac{3}{x+y}-\dfrac{2x}{(x+y)^2}$.

16. $\dfrac{3b}{(b+1)^2}-\dfrac{2}{b+1}$.

17. $\dfrac{2x+y}{x^2-y^2}-\dfrac{2x-y}{(x+y)^2}$.

18. $\dfrac{b+c}{b^2-2bc+c^2}-\dfrac{b-2c}{b^2-c^2}$.

19. $\dfrac{x}{xy-y^2}-\dfrac{xy}{x^3-x^2y}$.

20. $\dfrac{a^2+2a}{a^2+a-2}-\dfrac{a}{a+1}$.

21. $\dfrac{4a^2-b^2}{2ab-b^2}-\dfrac{4a}{2a+b}$.

22. $\dfrac{x^2}{x^3+1}-\dfrac{1}{x+1}$.

23. $\dfrac{2b-4}{b^3+8}+\dfrac{1}{b+2}$.

24. $\dfrac{x^2-2y^2}{x^2+xy+y^2}+\dfrac{x^2y^2-2y^4}{x^3y-y^4}$.

Find the value of

25. $\dfrac{1}{a^2-6a+9}-\dfrac{1}{a^2-5a+6}$.

26. $\dfrac{1}{a^2-3a+2}+\dfrac{1}{a^2+3a-10}$.

27. $x+2-\dfrac{x-2}{x-1}$.

28. $4+\dfrac{a-6}{2+a}-2a$.

29. $\dfrac{1}{x^2}+\dfrac{x^2}{x+1}-\dfrac{1}{x}$.

30. $\dfrac{1}{x}-\dfrac{2}{x-2}+\dfrac{1}{x-4}$.

31. $\dfrac{3}{a+2}-\dfrac{1}{a-2}-\dfrac{2}{a+6}$.

166. The following examples furnish additional practice in the simplification of fractions.

Example. Simplify $\dfrac{4}{3b+3}-\dfrac{2}{5b-5}+\dfrac{7b+5}{b^2-1}$.

$$\text{The expression}=\frac{4}{3(b+1)}-\frac{2}{5(b-1)}+\frac{7b+5}{b^2-1}$$

$$=\frac{20(b-1)-6(b+1)+7b+5}{15(b^2-1)}$$

$$=\frac{21b-21}{15(b^2-1)}=\frac{21(b-1)}{15(b^2-1)}=\frac{7}{5(b+1)}.$$

167. Sometimes the work will be simplified by combining two of the fractions together, instead of finding the lowest common multiple of all the denominators at once.

Example. Simplify $\dfrac{3}{8(a-x)}-\dfrac{1}{8(a+x)}-\dfrac{a-2x}{4(a^2+x^2)}$.

Taking the first two fractions together,

$$\text{the expression}=\frac{3(a+x)-(a-x)}{8(a^2-x^2)}-\frac{a-2x}{4(a^2+x^2)}$$

$$=\frac{a+2x}{4(a^2-x^2)}-\frac{a-2x}{4(a^2+x^2)}$$

$$=\frac{(a+2x)(a^2+x^2)-(a-2x)(a^2-x^2)}{4(a^4-x^4)}$$

$$=\frac{a^3+2a^2x+ax^2+2x^3-(a^3-2a^2x-ax^2+2x^3)}{4(a^4-x^4)}$$

$$=\frac{4a^2x+2ax^2}{4(a^4-x^4)}=\frac{ax(2a+x)}{2(a^4-x^4)}.$$

EXAMPLES XXI. c.

Find the value of

1. $\dfrac{6}{2x-1}-\dfrac{3}{2x+1}-\dfrac{2-3x}{4x^2-1}$.

2. $\dfrac{1}{2a+3c}-\dfrac{1}{2a-3c}+\dfrac{6c}{4a^2-9c^2}$.

3. $\dfrac{1+2a}{3-3a}-\dfrac{3a^2+2a}{2-2a^2}+1$.

4. $\dfrac{2x}{9-6x}+\dfrac{5}{6+4x}-\dfrac{4x^2-9x}{27-12x^2}$.

5. $\dfrac{1}{x-a}+\dfrac{2a}{(x-a)^2}+\dfrac{a^2}{(x-a)^3}$.

6. $\dfrac{2}{(a+1)^2}-\dfrac{a-3}{(a+1)^4}+\dfrac{2}{(a+1)^3}$.

7. $\dfrac{a^3}{(a-b)^3}-\dfrac{a}{a-b}-\dfrac{ab}{(a-b)^2}$.

8. $\dfrac{1}{x+3}-\dfrac{1}{x}+\dfrac{1}{x+1}$.

9. $\dfrac{1}{2y^2-y-3}-\dfrac{1}{2y^2+y-1}$.

10. $\dfrac{5}{4+3x-x^2}-\dfrac{2}{3+4x+x^2}$.

11. $\dfrac{1}{z(z-1)}+\dfrac{1}{z(z+1)}-\dfrac{2}{z^2-1}$.

12. $\dfrac{2}{(x-2)^2}-\dfrac{x}{x^2+4}+\dfrac{1}{x-2}$.

13. $\dfrac{2}{3-a}-\dfrac{3}{(2+a)(3-a)(1+2a)}+\dfrac{1}{(3-a)(1+2a)}$.

14. $\dfrac{y-2}{(y-3)(y-4)}-\dfrac{2(y-3)}{(y-2)(y-4)}+\dfrac{y-4}{(y-2)(y-3)}$.

15. $\dfrac{1}{1-x}-\dfrac{2+x}{(1-x)(2-x)}+\dfrac{2+3x+3x^2}{(1-x)(2-x)(3+x)}$.

16. $\dfrac{2}{x^2-5xy+6y^2}-\dfrac{3}{x^2-xy-6y^2}+\dfrac{1}{x^2-4y^2}$.

17. $\dfrac{5a}{6(a^2-1)}-\dfrac{a+3}{2(a^2+2a-3)}+\dfrac{a+1}{3a^2+6a+3}$.

18. $\dfrac{x-5}{x^2-4x-5}+\dfrac{2x}{x^2+2x}-\dfrac{3x-6}{x^2+x-6}$.

19. $\dfrac{a}{a-b}-\dfrac{b^2}{a^2+ab+b^2}-\dfrac{a^3+b^3}{a^3-b^3}$.

20. $\dfrac{3(6-x)}{x^3+27}+\dfrac{x-3}{x^2-3x+9}-\dfrac{1}{x+3}$.

21. $\dfrac{1}{(x-y)^2}-\dfrac{1}{x^2+2xy+y^2}-\dfrac{4xy}{x^4-2x^2y^2+y^4}$.

22. $\dfrac{x}{(x-a)^2}-\dfrac{a}{x^2-a^2}-\dfrac{ax}{(x-a)^3}$.

23. $\dfrac{1}{2-x}+\dfrac{1}{2+x}-\dfrac{3}{4+x^2}$.

24. $\dfrac{x}{4(1+x)}-\dfrac{x}{4(1-x)}+\dfrac{3}{2(1+x^2)}$.

25. $\dfrac{3}{2m-4}-\dfrac{3}{2m+4}-\dfrac{2}{3m^2+12}$.

26. $\dfrac{2a-6}{a^2-6a+9}-\dfrac{2a-3}{a^2-a-6}$.

27. $\dfrac{a}{a-b}-\dfrac{b}{a+b}-\dfrac{b^2}{a^2+b^2}$.

28. $\dfrac{x-3}{x-4}-\dfrac{x-1}{x-2}-\dfrac{1}{(x-2)^2}$.

29. $\dfrac{x-3}{x-6}-\dfrac{x-6}{x-3}+\dfrac{x-3}{x}-\dfrac{x}{x-3}$.

30. $\dfrac{1}{a-6}-\dfrac{1}{3(a-2)}+\dfrac{1}{3(a+2)}-\dfrac{1}{a+6}$.

168. To find a meaning for the fraction $\dfrac{-a}{-b}$, we define it as the quotient resulting from the division of $-a$ by $-b$; and this is obtained by dividing a by b, and, by the rule of signs, prefixing $+$.

Therefore $$\frac{-a}{-b}=+\frac{a}{b}=\frac{a}{b} \qquad\qquad (1).$$

Again, $\dfrac{-a}{b}$ is the quotient resulting from the division of $-a$ by b; and this is obtained by dividing a by b, and, by the rule of signs, prefixing $-$.

Therefore $$\frac{-a}{b}=-\frac{a}{b} \qquad\qquad (2).$$

Likewise $\dfrac{a}{-b}$ is the quotient resulting from the division of a by $-b$; and this is obtained by dividing a by b, and, by the rule of signs, prefixing $-$.

Therefore $$\frac{a}{-b}=-\frac{a}{b} \qquad\qquad (3).$$

These results may be enunciated as follows:

(1) *If the signs of both numerator and denominator of a fraction be changed, the sign of the whole fraction will be unchanged.*

(2) *If the sign of the numerator alone be changed, the sign of the whole fraction will be changed.*

(3) *If the sign of the denominator alone be changed, the sign of the whole fraction will be changed.*

Example 1. $\dfrac{b-a}{y-x}=\dfrac{-(b-a)}{-(y-x)}=\dfrac{-b+a}{-y+x}=\dfrac{a-b}{x-y}$.

Example 2. $\dfrac{x-x^2}{2y}=-\dfrac{-x+x^2}{2y}=-\dfrac{x^2-x}{2y}$.

Example 3. $\dfrac{3x}{4-x^2} = -\dfrac{3x}{-4+x^2} = -\dfrac{3x}{x^2-4}$.

Example 4. Simplify $\dfrac{a}{x+a} + \dfrac{2x}{x-a} + \dfrac{a(3x-a)}{a^2-x^2}$.

Here it is evident that the lowest common denominator of the first two fractions is $x^2 - a^2$, therefore it will be convenient to alter the sign of the denominator in the third fraction.

$$\text{Thus the expression} = \frac{a}{x+a} + \frac{2x}{x-a} - \frac{a(3x-a)}{x^2-a^2}$$
$$= \frac{a(x-a) + 2x(x+a) - a(3x-a)}{x^2-a^2}$$
$$= \frac{ax - a^2 + 2x^2 + 2ax - 3ax + a^2}{x^2-a^2}$$
$$= \frac{2x^2}{x^2-a^2}.$$

Example 5. Simplify $\dfrac{1}{(a-b)(a-c)} + \dfrac{1}{(b-c)(b-a)} + \dfrac{1}{(c-a)(c-b)}$.

Here in finding the L.C.M. of the denominators it must be observed that there are not *six* different compound factors to be considered; for three of them differ from the other three only in sign.

Thus
$$(a-c) = -(c-a),$$
$$(b-a) = -(a-b),$$
$$(c-b) = -(b-c).$$

Hence, replacing the second factor in each denominator by its equivalent, we may write the expression in the form

$$-\frac{1}{(a-b)(c-a)} - \frac{1}{(b-c)(a-b)} - \frac{1}{(c-a)(b-c)}.$$

Now the L.C.M. is $(b-c)(c-a)(a-b)$;

$$\text{and the expression} = \frac{-(b-c)-(c-a)-(a-b)}{(b-c)(c-a)(a-b)}$$
$$= \frac{-b+c-c+a-a+b}{(b-c)(c-a)(a-b)}$$
$$= 0.$$

Note. In examples of this kind it will be found convenient to arrange the expressions **cyclically**, that is, so that a is followed by b, b by c, and c by a.

169. If the sign of each of *two* factors in a product is changed, the sign of the product is unaltered; thus

$$(a-x)(b-x)=\{-(x-a)\}\{-(x-b)\}=(x-a)(x-b).$$

Similarly, $$(a-x)^2=(x-a)^2.$$

In other words, in the simplification of fractions we may change the sign of each of *two* factors in a denominator without altering the sign of the fraction; thus

$$\frac{1}{(b-a)(c-b)}=\frac{1}{(a-b)(b-c)}.$$

170. The arrangement adopted in the following example is worthy of notice.

Example. Simplify $\frac{1}{a-x}-\frac{1}{a+x}-\frac{2x}{a^2+x^2}-\frac{4x^3}{a^4+x^4}$.

Here it should be evident that the first two denominators give L.C.M. a^2-x^2, which readily combines with a^2+x^2 to give L.C.M. a^4-x^4, which again combines with a^4+x^4 to give L.C.M. a^8-x^8. Hence it will be convenient to proceed as follows:

$$\begin{aligned}\text{The expression} &= \frac{a+x-(a-x)}{a^2-x^2}-\ldots\ldots-\ldots\ldots\\ &=\frac{2x}{a^2-x^2}-\frac{2x}{a^2+x^2}-\ldots\ldots\\ &=\frac{4x^3}{a^4-x^4}-\frac{4x^3}{a^4+x^4}\\ &=\frac{8x^7}{a^8-x^8}.\end{aligned}$$

EXAMPLES XXI. d.

Find the value of

1. $\frac{5}{1+2x}-\frac{3x}{1-2x}+\frac{4-13x}{4x^2-1}$.

2. $\frac{10}{9-a^2}-\frac{2}{3+a}+\frac{1}{a-3}$.

3. $\frac{5a}{6(a^2-1)}+\frac{1}{2(1-a)}+\frac{1}{3(a+1)}$.

4. $\frac{2y}{2y-3}-\frac{5}{6y+9}+\frac{12y+8}{27-12y^2}$.

5. $\frac{x+a}{x-a}-\frac{x-a}{x+a}+\frac{4ax}{a^2-x^2}$.

6. $\frac{3-2c}{3+2c}+\frac{2c+3}{2c-3}+\frac{12}{4c^2-9}$.

Find the value of

7. $\dfrac{a}{a-b}-\dfrac{b}{a+b}+\dfrac{b}{b-a}$.

8. $\dfrac{a}{a^2-b^2}+\dfrac{b}{a^2+b^2}+\dfrac{a}{b^2-a^2}$.

9. $\dfrac{a}{x^2-x}+\dfrac{a}{x-x^3}-\dfrac{a}{x^2-1}$.

10. $\dfrac{x^6-x^3y^3}{y^6-x^6}+\dfrac{x^3y^3}{x^3y^3-y^6}$.

11. $\dfrac{1}{(y-2)(y-3)}+\dfrac{2}{(y-1)(3-y)}+\dfrac{1}{(y-1)(y-2)}$.

12. $\dfrac{a}{(x-a)(a-b)}-\dfrac{b}{(x-b)(a-b)}+\dfrac{x}{(a-x)(b-x)}$.

13. $\dfrac{2}{x-1}+\dfrac{3}{(1-x)^2}-\dfrac{1}{2x-1}$.

14. $\dfrac{1}{a-b}-\dfrac{a}{(a-b)^2}-\dfrac{ab}{(b-a)^3}$.

15. $\dfrac{a+c}{(a-b)(x-a)}-\dfrac{b+c}{(b-a)(b-x)}$.

16. $\dfrac{x-z}{(x-y)(a-x)}-\dfrac{y-z}{(y-x)(y-a)}$.

17. $\dfrac{a+b}{b}-\dfrac{2a}{a+b}+\dfrac{a^3-a^2b}{b(b^2-a^2)}$.

18. $\dfrac{a^2-ab}{b^2-ab}-\dfrac{a^2-b^2}{ab-a^2}+\dfrac{a}{b}$.

19. $\dfrac{1}{a+x}+\dfrac{1}{a-2x}-\dfrac{1}{x-a}+\dfrac{1}{2x+a}$.

20. $\dfrac{3}{a+x}-\dfrac{1}{3x+a}+\dfrac{3}{x-a}+\dfrac{1}{a-3x}$.

21. $\dfrac{x}{(x-y)(x-z)}+\dfrac{y}{(y-z)(y-x)}+\dfrac{z}{(z-x)(z-y)}$.

22. $\dfrac{a}{(b-c)(b-a)}+\dfrac{b}{(c-a)(c-b)}+\dfrac{c}{(a-b)(a-c)}$.

23. $\dfrac{y-z}{(x-y)(x-z)}+\dfrac{z-x}{(y-z)(y-x)}+\dfrac{x-y}{(z-x)(z-y)}$.

24. $\dfrac{1+p}{(p-q)(p-r)}+\dfrac{1+q}{(q-r)(q-p)}+\dfrac{1+r}{(r-p)(r-q)}$.

25. $\dfrac{1}{4(x+a)}-\dfrac{1}{4(a-x)}+\dfrac{x}{2(x^2-a^2)}+\dfrac{x^3}{a^4-x^4}$.

26. $\dfrac{1}{2a^3(a+x)}-\dfrac{1}{2a^3(x-a)}+\dfrac{1}{a^2(a^2+x^2)}+\dfrac{2a^4}{x^8-a^8}$.

27. $\dfrac{a}{a^2-b^2}-\dfrac{b}{a^2+b^2}+\dfrac{a^3+b^3}{b^4-a^4}+\dfrac{ab}{(a+b)(a^2+b^2)}$.

28. $\dfrac{1}{x-2}+\dfrac{2}{(2+x)^2}+\dfrac{2}{(2-x)^2}-\dfrac{1}{x+2}$.

CHAPTER XXII.

MISCELLANEOUS FRACTIONS.

171. DEFINITION. A fraction whose numerator and denominator are whole numbers is called a **Simple Fraction.**

A fraction of which the numerator or denominator is itself a fraction is called a **Complex Fraction.**

Thus $\dfrac{\frac{a}{b}}{c}$, $\dfrac{\frac{a}{b}}{x}$, $\dfrac{\frac{a}{b}}{\frac{c}{d}}$ are Complex Fractions.

In the last of these types the outside quantities, a and d, are sometimes referred to as the *extremes*, while the two middle quantities, b and c, are called the *means*.

Instead of using the horizontal line to separate numerator and denominator, it is sometimes convenient to write complex fractions in the forms

$$a\Big/\frac{b}{c},\quad \frac{a}{b}\Big/x,\quad \frac{a}{b}\Big/\frac{c}{d}.$$

Simplification of Complex Fractions.

172. It is proved in the *Elementary Algebra*, Art. 111, that

$$\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}.$$

The student should notice the following particular cases, and should be able to write down the results readily.

$$\frac{1}{\frac{a}{b}}=1\div\frac{a}{b}=1\times\frac{b}{a}=\frac{b}{a}.$$

$$\frac{a}{\frac{1}{b}}=a\div\frac{1}{b}=a\times b=ab.$$

173. The following examples illustrate the simplification of complex fractions.

Example 1. Simplify $\dfrac{x+\dfrac{a^2}{x}}{x-\dfrac{a^4}{x^3}}$.

$$\text{The expression} = \left(x+\frac{a^2}{x}\right) \div \left(x-\frac{a^4}{x^3}\right) = \frac{x^2+a^2}{x} \div \frac{x^4-a^4}{x^3}$$

$$= \frac{x^2+a^2}{x} \times \frac{x^3}{x^4-a^4} = \frac{x^2}{x^2-a^2}.$$

Example 2. Simplify $\dfrac{\dfrac{3}{a}+\dfrac{a}{3}-2}{\dfrac{a}{6}+\dfrac{1}{2}-\dfrac{3}{a}}$.

Here the reduction may be simply effected by multiplying the fractions above and below by $6a$, which is the L.C.M. of the denominators.

$$\text{Thus the expression} = \frac{18+2a^2-12a}{a^2+3a-18}$$

$$= \frac{2(a^2-6a+9)}{(a+6)(a-3)} = \frac{2(a-3)}{a+6}.$$

Example 3. Simplify $\dfrac{\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a^2-b^2}{a^2+b^2}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}$.

$$\text{The numerator} = \frac{(a^2+b^2)^2-(a^2-b^2)^2}{(a^2+b^2)(a^2-b^2)} = \frac{4a^2b^2}{(a^2+b^2)(a^2-b^2)};$$

$$\text{similarly the denominator} = \frac{4ab}{(a+b)(a-b)}.$$

$$\text{Hence the fraction} = \frac{4a^2b^2}{(a^2+b^2)(a^2-b^2)} \div \frac{4ab}{(a+b)(a-b)}$$

$$= \frac{4a^2b^2}{(a^2+b^2)(a^2-b^2)} \times \frac{(a+b)(a-b)}{4ab}$$

$$= \frac{ab}{a^2+b^2}.$$

Note. To ensure accuracy and neatness, when the numerator and denominator are somewhat complicated, the beginner is advised to simplify each separately as in the above example.

174. In the case of **Continued Fractions** we begin from the lowest fraction, and simplify step by step.

Example. Find the value of $\dfrac{1}{4-\dfrac{3}{2+\dfrac{x}{1-x}}}$.

$$\text{The expression} = \frac{1}{4-\dfrac{3}{\dfrac{2-2x+x}{1-x}}} = \frac{1}{4-\dfrac{3(1-x)}{2-x}}$$

$$= \frac{1}{\dfrac{8-4x-3+3x}{2-x}} = \frac{1}{\dfrac{5-x}{2-x}}$$

$$= \frac{2-x}{5-x}.$$

EXAMPLES XXII. a.

Find the value of

1. $\dfrac{1}{x+\dfrac{y}{z}}$.

2. $\dfrac{a}{b-\dfrac{c}{d}}$.

3. $\dfrac{1-a}{\dfrac{1}{a^2}-1}$.

4. $\dfrac{b}{\dfrac{1}{1-a}}$.

5. $\dfrac{\dfrac{a}{x}-\dfrac{x}{a}}{\dfrac{1}{x}-\dfrac{1}{a}}$.

6. $\dfrac{\dfrac{1}{x^2}-\dfrac{1}{y^2}}{\dfrac{y}{x}-\dfrac{x}{y}}$.

7. $\dfrac{a-\dfrac{b}{d}}{\dfrac{a}{b}-\dfrac{1}{d}}$.

8. $\dfrac{\dfrac{p}{q}-r}{\dfrac{1}{pq}-\dfrac{r}{p^2}}$.

9. $\dfrac{a+\dfrac{6}{a}-5}{1+\dfrac{8}{a^2}-\dfrac{6}{a}}$.

10. $\dfrac{y-3+\dfrac{y^2}{3}}{y-\dfrac{9}{y}+3}$.

11. $\dfrac{\dfrac{1}{n}-\dfrac{3}{n^2}-\dfrac{4}{n^3}}{n-\dfrac{16}{n}}$.

12. $\dfrac{x-2+\dfrac{6}{x+3}}{x-4+\dfrac{12}{x+3}}$.

13. $\dfrac{b-2-\dfrac{6}{b+3}}{b-4+\dfrac{6}{b+3}}$.

14. $\dfrac{\dfrac{a}{b^2}-\dfrac{b}{a^2}}{\dfrac{1}{a^2}+\dfrac{1}{ab}+\dfrac{1}{b^2}}$.

15. $\dfrac{\dfrac{c+d}{c-d}-\dfrac{c-d}{c+d}}{\dfrac{c+d}{c-d}+\dfrac{c-d}{c+d}}$.

16. $\dfrac{a-\dfrac{a-b}{1-ab}}{1-\dfrac{a(a-b)}{1-ab}}$.

17. $\dfrac{\dfrac{x+3}{7}-\dfrac{x+3}{x+4}}{\dfrac{x-3}{4}+\dfrac{x-3}{x-1}}$.

18. $1+\dfrac{1}{1+\dfrac{1}{a}}$.

19. $x+\dfrac{1}{x-\dfrac{1}{x}}$.

20. $2-\dfrac{3}{4-\dfrac{c}{d}}$.

Find the value of

21. $\dfrac{x}{1+\dfrac{x}{1-\dfrac{1}{x}}}$.

22. $\dfrac{1}{x+\dfrac{1}{x+\dfrac{2}{x}}}$.

23. $\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{y}}}$.

24. $\dfrac{1-x^4}{2-\dfrac{x}{1-\dfrac{1}{1+x}}}$.

25. $\dfrac{y}{1-\dfrac{1-y}{1-\dfrac{y^2}{2-y}}}$.

26. $\dfrac{a}{b-\dfrac{c}{d-\dfrac{e}{f}}}$.

27. $\dfrac{x^2-1}{2x^2-\dfrac{4x^2-1}{1+\dfrac{x}{x-1}}}$.

28. $\dfrac{3a-2c}{3a-2c-\dfrac{3a}{1-\dfrac{3(a-c)}{3a-2c}}}$.

175. Sometimes it is convenient to express a single fraction as a group of fractions.

Example. $\dfrac{5x^2y-10xy^2+15y^3}{10x^2y^2}=\dfrac{5x^2y}{10x^2y^2}-\dfrac{10xy^2}{10x^2y^2}+\dfrac{15y^3}{10x^2y^2}$

$$=\frac{1}{2y}-\frac{1}{x}+\frac{3y}{2x^2}.$$

176. Since a fraction represents the quotient of the numerator by the denominator, we may often express a fraction in an equivalent form, partly integral and partly fractional.

Example 1. $\dfrac{x+7}{x+2}=\dfrac{(x+2)+5}{x+2}=1+\dfrac{5}{x+2}$.

Example 2. $\dfrac{3x-2}{x+5}=\dfrac{3(x+5)-15-2}{x+5}=\dfrac{3(x+5)-17}{x+5}=3-\dfrac{17}{x+5}$.

Example 3. Shew that $\dfrac{2x^2-7x-1}{x-3}=2x-1-\dfrac{4}{x-3}$.

By actual division,

$$\begin{array}{r|l} x-3 \;)\; 2x^2-7x-1 & (\,2x-1 \\ 2x^2-6x & \\ \hline -\;x-1 & \\ -\;x+3 & \\ \hline -4 & \end{array}$$

Thus the quotient is $2x-1$, and the remainder -4.

Therefore $\dfrac{2x^2-7x-1}{x-3}=2x-1-\dfrac{4}{x-3}$.

177. If the numerator be of lower dimensions than the denominator, we may still perform the division, and express the result in a form which is partly integral and partly fractional.

Example. Prove that $\dfrac{2x}{1+3x^2} = 2x - 6x^3 + 18x^5 - \dfrac{54x^7}{1+3x^2}$.

By division

$$\begin{array}{l} 1+3x^2\,)\,2x \qquad (\,2x - 6x^3 + 18x^5 \\ \quad\; \underline{2x + 6x^3} \\ \qquad -6x^3 \\ \qquad \underline{-6x^3 - 18x^5} \\ \qquad\qquad 18x^5 \\ \qquad\qquad \underline{18x^5 + 54x^7} \\ \qquad\qquad\quad -54x^7 \end{array}$$

whence the result follows.

Here the division may be carried on to any number of terms in the quotient, and we can stop at any term we please by taking for our remainder the fraction whose numerator is the remainder last found, and whose denominator is the divisor.

Thus, if we carried on the quotient to four terms, we should have

$$\frac{2x}{1+3x^2} = 2x - 6x^3 + 18x^5 - 54x^7 + \frac{162x^9}{1+3x^2}.$$

The terms in the quotient may be fractional; thus if x^2 is divided by $x^3 - a^3$, the first four terms of the quotient are $\dfrac{1}{x} + \dfrac{a^3}{x^4} + \dfrac{a^6}{x^7} + \dfrac{a^9}{x^{10}}$, and the remainder is $\dfrac{a^{12}}{x^{10}}$.

178. The following exercise contains miscellaneous examples which illustrate most of the processes connected with fractions.

EXAMPLES XXII. b.

Simplify the following fractions :

1. $\dfrac{1-x^3}{1+2x+2x^2+x^3}$.

2. $\dfrac{12x^2+x-1}{1-8x+16x^2} \div \dfrac{1+6x+9x^2}{16x^2-1}$.

3. $\dfrac{a+b}{a-b} + \dfrac{4ab}{b^2-a^2}$.

4. $\dfrac{a+b}{a^2-ab-2b^2} - \dfrac{2a}{a^2-4b^2}$.

5. $\dfrac{x^3-1}{x-1} - \dfrac{x^4+x^2+1}{x^2+x+1}$.

6. $\dfrac{(x+y)^2}{x-y} - \dfrac{(x-y)^2}{x+y}$.

Simplify the following fractions:

7. $\dfrac{abx^2-acx+bxy-cy}{ax^2+xy-ax-y}$.

8. $\dfrac{1}{x}\left(\dfrac{a}{a-x}-\dfrac{a}{a+3x}\right)-\dfrac{3}{a+3x}$.

9. $\dfrac{2}{x^2-6x+8}+\dfrac{3}{x^2-11x+28}+\dfrac{5}{x^2-9x+14}$.

10. $\dfrac{3a-\dfrac{1}{3a}}{3a+1}\times\dfrac{a}{3a-1}$.

11. $\dfrac{\dfrac{x^2+a^2}{x}-a}{a^3+x^3}+\dfrac{\dfrac{1}{x}}{x+a}$.

12. $\dfrac{1}{1-\dfrac{x}{x-1}}-\dfrac{1}{\dfrac{x}{x+1}-1}$.

13. $\dfrac{2x^2-\dfrac{7x^2-27}{x-1}}{3x^2-\dfrac{3(x^2-27x+54)}{1-x}}$.

14. $\dfrac{cd(a^2+b^2)+ab(c^2+d^2)}{cd(a^2-b^2)+ab(c^2-d^2)}$.

15. $\left(\dfrac{x}{1+x^2}\times\dfrac{1+x}{x^2}\right)-\dfrac{1}{x^2}$.

16. $\dfrac{1}{x^3-3ax^2+4a^3}-\dfrac{1}{x^3-ax^2-4a^2x+4a^3}$.

17. $\dfrac{\dfrac{a^2+4}{2}-a}{\dfrac{2}{a}-1}\times\dfrac{a^2-4}{a^3+8}$.

18. $\dfrac{\dfrac{2}{a^2}\left(4a^2-\dfrac{1}{9}\right)}{\dfrac{1}{a}+6}+\dfrac{1}{3}$.

19. $\dfrac{2x^3+x^2-3x}{35x^2+24x-35}\times\dfrac{5x^2-8x-21}{x^3+7x^2-8x}\div\dfrac{2x^2-3x-9}{7x^2+51x-40}$.

20. $\dfrac{q+r-p}{(p-q)(p-r)}+\dfrac{r+p-q}{(q-r)(q-p)}+\dfrac{p+q-r}{(r-p)(r-q)}$.

21. $\left\{\left(\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}\right)\div\left(\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}\right)\right\}-\dfrac{x^3+x^2y+xy^2+y^3}{2x^2y+2xy^2}$.

22. $\dfrac{a^2-(b-c)^2}{(c+a)^2-b^2}+\dfrac{b^2-(c-a)^2}{(a+b)^2-c^2}+\dfrac{c^2-(a-b)^2}{(b+c)^2-a^2}$.

23. $\left(\dfrac{x^2}{y}+\dfrac{y^2}{x}\right)\left(\dfrac{1}{y^2-x^2}\right)-\dfrac{y}{x^2+xy}+\dfrac{x}{xy-y^2}-\dfrac{1}{x+y}$.

24. $\dfrac{a^3-1}{a^2+a-6}\div\left[\dfrac{a^2-4a+3}{a^2-4a+4}\div\left\{\dfrac{a^2-9}{a^4+a^2+1}\div\dfrac{a^2-a-2}{a^3+1}\right\}\right]$.

25. $\left(\dfrac{1}{2a}+\dfrac{1}{2a-x}\right)\left(\dfrac{1}{3a}-\dfrac{1}{3a-x}\right)-\dfrac{x^2-4ax}{6a^2(x-2a)(x-3a)}$.

Simplify the following fractions:

26. $\dfrac{1}{6a-6}-\dfrac{1}{6a+6}+\dfrac{1}{3a^2+3}-\dfrac{2a^2}{3a^4+3}.$

27. $\dfrac{4ab^2}{2a^4+32b^4}+\dfrac{1}{8a+16b}-\dfrac{a}{4a^2+16b^2}-\dfrac{1}{8(2b-a)}.$

28. $\dfrac{3b^2+b}{6b^2-1-b}+\dfrac{2b-7}{1-2b}+\dfrac{2b^2-3b}{4b^2-8b+3}+3.$

29. $\dfrac{1}{\left(1-\dfrac{y}{x}\right)\left(1-\dfrac{z}{x}\right)}+\dfrac{1}{\left(1-\dfrac{z}{y}\right)\left(1-\dfrac{x}{y}\right)}+\dfrac{1}{\left(1-\dfrac{x}{z}\right)\left(1-\dfrac{y}{z}\right)}.$

30. $\dfrac{\dfrac{x^4+x^3y+x^2y^2}{(x^2-y^2)^3}\times\left(1+\dfrac{y}{x}\right)^2}{\left(1-\dfrac{y^3}{x^3}\right)\div\left(\dfrac{y}{x^2}+\dfrac{1}{x}\right)}.$

31. $\dfrac{m^2+\dfrac{1}{m^2}+1}{m^2-\dfrac{1}{m^4}}-\dfrac{m^3+m}{\dfrac{1}{m}-m^3}.$

32. $\left(\dfrac{ab}{ab-b^2}-\dfrac{ac}{ac-bc}\right)\left(\dfrac{1}{1-\dfrac{b}{a}}+\dfrac{1}{1-\dfrac{c}{b}}+\dfrac{1}{1-\dfrac{a}{c}}\right).$

33. $\left\{\dfrac{\dfrac{a-c}{1+ac}+c}{1-\dfrac{c(a-c)}{1+ac}}-\dfrac{a-\dfrac{a-c}{1-ac}}{1-\dfrac{a(a-c)}{1-ac}}\right\}\div\left(\dfrac{a}{c}-\dfrac{c}{a}\right).$

34. $\dfrac{\dfrac{1}{(3a+x)^2}}{\dfrac{1}{a}}+\dfrac{\dfrac{1}{3}}{x-3a}-\dfrac{1}{3(x+3a)}+\dfrac{1}{\dfrac{(x-3a)^2}{a}}.$

CHAPTER XXIII.

HARDER EQUATIONS.

179. SOME of the equations in this chapter will serve as a useful exercise for revision of the methods already explained; but we also add others presenting more difficulty, the solution of which will often be facilitated by some special artifice.

The following examples worked in full will sufficiently illustrate the most useful methods.

Example 1. Solve $\dfrac{6x-3}{2x+7}=\dfrac{3x-2}{x+5}$.

Clearing of fractions, we have

$$(6x-3)(x+5)=(3x-2)(2x+7),$$
$$6x^2+27x-15=6x^2+17x-14;$$
$$\therefore \quad 10x=1;$$
$$\therefore \quad x=\frac{1}{10}.$$

Note. By a simple reduction many equations can be brought to the form in which the above equation is given. When this is the case, the necessary simplification is readily completed by multiplying across or "multiplying up," as it is sometimes called.

Example 2. Solve $\dfrac{8x+23}{20}-\dfrac{5x+2}{3x+4}=\dfrac{2x+3}{5}-1$.

Multiplying by 20, we have

$$8x+23-\frac{20(5x+2)}{3x+4}=8x+12-20.$$

By transposition, $\qquad 31=\dfrac{20(5x+2)}{3x+4}.$

Multiplying across, $\qquad 93x+124=20(5x+2),$

$$84=7x;$$
$$\therefore \quad x=12.$$

180. When two or more fractions have the same denominator, they should be taken together and simplified.

Example 1. Solve $\frac{24-5x}{x-2}+\frac{8x-49}{4-x}=\frac{28}{x-2}-13.$

By transposition, we have

$$\frac{8x-49}{4-x}+13=\frac{28-(24-5x)}{x-2};$$

$$\therefore \quad \frac{3-5x}{4-x}=\frac{4+5x}{x-2}.$$

Multiplying across, we have

$$3x-5x^2-6+10x=16-4x+20x-5x^2;$$

that is,

$$-3x=22;$$

$$\therefore \quad x=-\frac{22}{3}.$$

Example 2. Solve $\frac{x-8}{x-10}+\frac{x-4}{x-6}=\frac{x-5}{x-7}+\frac{x-7}{x-9}.$

This equation might be solved by at once clearing of fractions, but the work would be laborious. The solution will be much simplified by proceeding as follows.

The equation may be written in the form

$$\frac{(x-10)+2}{x-10}+\frac{(x-6)+2}{x-6}=\frac{(x-7)+2}{x-7}+\frac{(x-9)+2}{x-9};$$

whence we have

$$1+\frac{2}{x-10}+1+\frac{2}{x-6}=1+\frac{2}{x-7}+1+\frac{2}{x-9};$$

which gives

$$\frac{1}{x-10}+\frac{1}{x-6}=\frac{1}{x-7}+\frac{1}{x-9}.$$

Transposing,

$$\frac{1}{x-10}-\frac{1}{x-7}=\frac{1}{x-9}-\frac{1}{x-6};$$

$$\therefore \quad \frac{3}{(x-10)(x-7)}=\frac{3}{(x-9)(x-6)}.$$

Hence, since the numerators are equal, the denominators must be equal;

that is,

$$(x-10)(x-7)=(x-9)(x-6),$$

$$x^2-17x+70=x^2-15x+54;$$

$$\therefore \quad 16=2x;$$

$$\therefore \quad x=8.$$

EXAMPLES XXIII. a.

Solve the following equations :

1. $\dfrac{3}{5x-9}=\dfrac{1}{4x-10}$.

2. $\dfrac{7}{6x-17}=\dfrac{3}{4x-13}$.

3. $\dfrac{7}{9}=\dfrac{3-4x}{4-5x}$.

4. $\dfrac{1}{6-5x}+\dfrac{4}{17x+3}=0$.

5. $\dfrac{5x-8}{x-4}=\dfrac{5x+14}{x+7}$.

6. $\dfrac{8x-1}{6x+2}=\dfrac{4x-3}{3x-1}$.

7. $\dfrac{22x-12}{8x-5}=2+\dfrac{3x+7}{4x+8}$.

8. $\dfrac{9x-22}{2x-5}-\dfrac{3x-5}{2x-7}=3$.

9. $\dfrac{8x-19}{4x-10}-\dfrac{1}{2}=\dfrac{3x-4}{2x+1}$.

10. $\dfrac{7x+2}{3(x-1)}=\dfrac{1}{3}+\dfrac{6x-1}{3x+1}$.

11. $\dfrac{x-5}{2}+\dfrac{2x-1}{3x+2}=\dfrac{5x-1}{10}-1\dfrac{2}{5}$.

12. $\dfrac{5x-17}{13-4x}+\dfrac{2x-11}{14}-\dfrac{23}{42}=\dfrac{3x-7}{21}$.

13. $x-\dfrac{4x-3}{7x+4}-\dfrac{1-9x}{6}=\dfrac{4x+3}{8}-\dfrac{1}{24}+2x$.

14. $\dfrac{3}{x+1}-\dfrac{2\frac{1}{3}}{x+2}=\dfrac{1}{x+3}-\dfrac{1}{3x+6}$.

15. $\dfrac{3\frac{1}{2}}{x-4}-\dfrac{18}{3x-18}=\dfrac{7}{4x-16}-\dfrac{4}{x-6}$.

16. $\dfrac{1}{x+6}+\dfrac{1}{3x+12}=\dfrac{3}{2x+10}-\dfrac{1}{6(x+4)}$.

17. $\dfrac{x-1}{x-2}-\dfrac{x-5}{x-6}=\dfrac{x-3}{x-4}-\dfrac{x-7}{x-8}$.

18. $\dfrac{1}{x-9}+\dfrac{1}{x-17}=\dfrac{1}{x-11}+\dfrac{1}{x-15}$.

19. $\dfrac{1}{2x-1}+\dfrac{1}{2x-7}=\dfrac{1}{2x-3}+\dfrac{1}{2x-5}$.

20. $\dfrac{x-1}{x-2}-\dfrac{x}{x-1}=\dfrac{x-4}{x-5}-\dfrac{x-3}{x-4}$.

21. $\dfrac{5x-64}{x-13}-\dfrac{4x-55}{x-14}=\dfrac{2x-11}{x-6}-\dfrac{x-6}{x-7}$.

Solve the following equations:

22. $\dfrac{5x+31}{x+6}-\dfrac{2x+9}{x+5}=\dfrac{x-6}{x-5}+\dfrac{2x-13}{x-6}.$

23. $\dfrac{12x+1}{3x-1}+\dfrac{5}{1-9x^2}=\dfrac{11+12x}{1+3x}.$

24. $\dfrac{5x^2}{x^2-9}-\dfrac{x+3}{x-3}=5-\dfrac{x-3}{x+3}.$

[For additional examples see *Elementary Algebra*.]

Literal Equations.

181. In the equations we have discussed hitherto the coefficients have been numerical quantities. When equations involve *literal* coefficients, these are supposed to be known, and will appear in the solution.

Example 1. Solve $(x+a)(x+b)-c(a+c)=(x-c)(x+c)+ab.$

Multiplying out, we have

$$x^2+ax+bx+ab-ac-c^2=x^2-c^2+ab;$$

whence

$$ax+bx=ac,$$

$$(a+b)x=ac;$$

$$\therefore\quad x=\frac{ac}{a+b}.$$

Example 2. Solve $\dfrac{a}{x-a}-\dfrac{b}{x-b}=\dfrac{a-b}{x-c}.$

Simplifying the left side, we have

$$\frac{a(x-b)-b(x-a)}{(x-a)(x-b)}=\frac{a-b}{x-c},$$

$$\frac{(a-b)x}{(x-a)(x-b)}=\frac{a-b}{x-c};$$

$$\therefore\quad \frac{x}{(x-a)(x-b)}=\frac{1}{x-c}.$$

Multiplying across, $x^2-cx=x^2-ax-bx+ab,$

$$ax+bx-cx=ab,$$

$$(a+b-c)x=ab;$$

$$\therefore\quad x=\frac{ab}{a+b-c}.$$

Example 3. Solve the simultaneous equations :

$$ax - by = c \quad \dots\dots (1),$$
$$px + qy = r \quad \dots\dots (2).$$

To eliminate y, multiply (1) by q and (2) by b;

thus
$$aqx - bqy = cq,$$
$$bpx + bqy = br.$$

By addition,
$$(aq + bp)x = cq + br;$$
$$\therefore \quad x = \frac{cq + br}{aq + bp}.$$

We might obtain y by substituting this value of x in *either* of the equations (1) or (2); but y is more conveniently found by eliminating x, as follows.

Multiplying (1) by p and (2) by a, we have

$$apx - bpy = cp,$$
$$apx + aqy = ar.$$

By subtraction,
$$(aq + bp)y = ar - cp;$$
$$\therefore \quad y = \frac{ar - cp}{aq + bp}.$$

EXAMPLES XXIII. b.

Solve the following equations :

1. $ax + b^2 = a^2 - bx.$

2. $x^2 - a^2 = (2a - x)^2.$

3. $a^2(a - x) + abx = b^2(x - b).$

4. $(b + 1)(x + a) = (b - 1)(x - a)$

5. $a(x + b) - b^2 = a^2 - b(a - x).$

6. $c^2x - d^3 = d^2x + c^3.$

7. $a(x - a) + b(x - b) + c(x - c) = 2(ab + bc + ca).$

8. $\dfrac{a^2}{x} - b = \dfrac{b^2}{x} + a.$

9. $\dfrac{x}{2a} = \dfrac{x}{b} + \dfrac{1}{b^2} - \dfrac{1}{4a^2}.$

10. $x + (x - a)(x - b) + a^2 + b^2 = b + x^2 - a(b - 1).$

11. $\dfrac{2x - a}{b} - \dfrac{3x - b}{a} = \dfrac{3a^2 - 8b^2}{ab}.$

12. $\dfrac{a - x}{a - b} - \dfrac{b - x}{a + b} = \dfrac{a^2 + b^2}{a^2 - b^2}.$

13. $\dfrac{ax - b}{c} + \dfrac{bx - c}{a} = \dfrac{a - cx}{b}.$

14. $\dfrac{x + a - b}{x + b + c} = \dfrac{x + b - c}{x + a + b}.$

15. $p(p - x) - \dfrac{p}{q}(x - q)^2 - p(p - q) + pq\left(\dfrac{x}{q} - 1\right)^2 = 0.$

Solve the following simultaneous equations :

16. $x-y=a+b$,
$ax+by=0$.

17. $cx-dy=c^2+d^2$,
$x+y=2c$.

18. $ax=by$,
$x-y=c$.

19. $\frac{x}{2}+\frac{y}{3}=a+b$,
$\frac{x}{a}+\frac{y}{b}=5$.

20. $\frac{a}{x}-\frac{b}{y}=0$,
$\frac{x}{a}+\frac{y}{b}=2$.

21. $\frac{x+y}{x-y}=\frac{a}{b}$,
$\frac{x-y}{a+b}=2b$

22. $\frac{x}{b}-\frac{y}{a}=\frac{a}{b}+\frac{b}{a}$,
$a(a+x)=b(b-y)$.

23. $\frac{x+y}{p}-\frac{x-y}{q}=0$.
$\frac{x-y}{2p}+\frac{x+y}{2q}=p^2+q^2$.

24. $\frac{2x-b}{a}=\frac{2y+a}{b}=\frac{3x+y}{a+2b}$.

25. $\frac{ax+by}{bx+ay}=\frac{1}{2}=\frac{a^2-b^2}{bx+ay}$.

Irrational or Surd Equations.

182. DEFINITION. If the root of a quantity cannot be exactly obtained the indicated root is called a **surd**.

Thus $\sqrt{2}$, $\sqrt[3]{5}$, $\sqrt[5]{a^3}$, $\sqrt{a^2+b^2}$ are surds.

A surd is sometimes called an **irrational quantity**; and quantities which are not surds are, for the sake of distinction, termed **rational quantities.**

183. Sometimes equations are proposed in which the unknown quantity appears under the radical sign. For a fuller discussion of surd equations the student may consult the *Elementary Algebra*, Chap. XXXII. Here we shall only consider a few simple cases, which can generally be solved by the following method. Bring to one side of the equation a single radical term by itself: on squaring both sides this radical will disappear. By repeating this process any remaining radicals can in turn be removed.

Example 1. Solve $2\sqrt{x}-\sqrt{4x-11}=1$.

Transposing, $2\sqrt{x}-1=\sqrt{4x-11}$.

Square both sides; then $4x-4\sqrt{x}+1=4x-11$,

$$4\sqrt{x}=12,$$
$$\sqrt{x}=3;$$
$$\therefore\ x=9.$$

Example 2. Solve $$2+\sqrt[3]{x-5}=13.$$

Transposing, $$\sqrt[3]{x-5}=11.$$

Here we must *cube* both sides; thus $x-5=1331$;

whence $$x=1336.$$

Example 3. Solve $\dfrac{6\sqrt{x}-11}{3\sqrt{x}}=\dfrac{2\sqrt{x}+1}{\sqrt{x}+6}$.

Multiplying across, we have

$$(6\sqrt{x}-11)(\sqrt{x}+6)=3\sqrt{x}(2\sqrt{x}+1);$$

that is, $$6x-11\sqrt{x}+36\sqrt{x}-66=6x+3\sqrt{x},$$

$$-11\sqrt{x}+36\sqrt{x}-3\sqrt{x}=66,$$

$$22\sqrt{x}=66,$$

$$\sqrt{x}=3;$$

$$\therefore\quad x=9.$$

EXAMPLES XXIII. c.

Solve the equations:

1. $\sqrt{x-2}=1.$

2. $\sqrt{5-2x}=7.$

3. $\sqrt[3]{x-7}=2.$

4. $2\sqrt{x+1}=3.$

5. $3\sqrt[3]{1-2x}=-1.$

6. $\dfrac{1}{2}=\sqrt[3]{2x}.$

7. $\sqrt{1-5x}=3\sqrt{1-x}.$

8. $2\sqrt{5x-3}-7\sqrt{x}=0.$

9. $\sqrt{4x^2-11x-7}=2x-3.$

10. $3\sqrt{1-7x+4x^2}=5-6x.$

11. $1+\sqrt[3]{x^3-3x^2+7x-11}=x.$

12. $\sqrt{x-11}=\sqrt{x}-1.$

13. $\sqrt{4x+13}+2\sqrt{x}=13.$

14. $3+\sqrt{12x-33}=2\sqrt{3x}.$

15. $\dfrac{\sqrt{x}-1}{\sqrt{x}+3}=\dfrac{\sqrt{x}-3}{\sqrt{x}}.$

16. $\dfrac{\sqrt{x}+4}{3\sqrt{x}-8}=\dfrac{\sqrt{x}+5}{3\sqrt{x}-7}.$

17. $\dfrac{2\sqrt{x}-3}{\sqrt{x}-1}=\dfrac{2\sqrt{x}-\frac{1}{3}}{\sqrt{x}+\frac{2}{3}}.$

18. $\dfrac{2\sqrt{x}-7}{\sqrt{x}-\frac{14}{3}}=2+\dfrac{15}{4\sqrt{x}-13}.$

19. $\sqrt{1+4x}+2\sqrt{x}=\dfrac{3}{\sqrt{x}}.$

20. $\sqrt{x}+\sqrt{x-3}=\dfrac{1}{\sqrt{x-3}}.$

21. $\sqrt{4x+7}-\sqrt{x+1}=\sqrt{x-3}.$

22. $\sqrt{4x-3}-\sqrt{x+3}=\sqrt{x-4}.$

CHAPTER XXIV.

HARDER PROBLEMS.

184. IN previous chapters we have given collections of problems which lead to simple equations. We add here a few examples of somewhat greater difficulty.

Example 1. If the numerator of a fraction is increased by 2 and the denominator by 1, it becomes equal to $\frac{5}{8}$; and if the numerator and denominator are each diminished by 1, it becomes equal to $\frac{1}{2}$: find the fraction.

Let x be the numerator of the fraction, y the denominator; then the fraction is $\dfrac{x}{y}$.

From the first supposition,

$$\frac{x+2}{y+1}=\frac{5}{8} \quad\text{.................................}\quad (1),$$

from the second,

$$\frac{x-1}{y-1}=\frac{1}{2} \quad\text{.................................}\quad (2).$$

From the first equation, $\quad 8x-5y=-11$,
and from the second, $\quad 2x-\ y=1$;
whence $\quad x=8,\ y=15$.

Thus the fraction is $\dfrac{8}{15}$.

Example 2. At what time between 4 and 5 o'clock will the minute-hand of a watch be 13 minutes in advance of the hour-hand?

Let x denote the required number of minutes after 4 o'clock; then, as the minute-hand travels twelve times as fast as the hour-hand, the hour-hand will move over $\dfrac{x}{12}$ minute-divisions in x minutes. At 4 o'clock the minute-hand is 20 divisions behind the hour-hand, and finally the minute-hand is 13 divisions in advance; therefore the minute-hand moves over $20+13$, or 33 divisions more than the hour-hand.

Hence $$x = \frac{x}{12} + 33,$$

$$\frac{11}{12}x = 33;$$

$$\therefore \quad x = 36.$$

Thus the time is 36 minutes past 4.

If the question be asked as follows: "At what *times* between 4 and 5 o'clock will there be 13 minutes between the two hands?" we must also take into consideration the case when the minute-hand is 13 divisions *behind* the hour-hand. In this case the minute-hand gains 20 – 13, or 7 divisions.

Hence $$x = \frac{x}{12} + 7,$$

which gives $$x = 7\frac{7}{11}.$$

Therefore the *times* are $7\frac{7}{11}'$ past 4, and 36′ past 4.

Example 3. A grocer buys 15 lbs. of figs and 28 lbs. of currants for \$2.60; by selling the figs at a loss of 10 per cent., and the currants at a gain of 30 per cent., he clears 30 cents on his outlay; how much per pound did he pay for each?

Let x, y denote the number of cents in the price of a pound of figs and currants respectively; then the outlay is

$$15x + 28y \text{ cents.}$$

Therefore $$15x + 28y = 260 \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The loss upon the figs is $\frac{1}{10} \times 15x$ cents, and the gain upon the currants is $\frac{3}{10} \times 28y$ cents; therefore the total gain is

$$\frac{42y}{5} - \frac{3x}{2} \text{ cents;}$$

$$\therefore \frac{42y}{5} - \frac{3x}{2} = 30;$$

that is, $$28y - 5x = 100 \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

From (1) and (2) we find that $x=8$, and $y=5$; that is, the figs cost 8 cents a pound, and the currants cost 5 cents a pound.

Example 4. Two persons A and B start simultaneously from two places, c miles apart, and walk in the same direction. A travels at the rate of p miles an hour, and B at the rate of q miles; how far will A have walked before he overtakes B?

Suppose A has walked x miles, then B has walked $x-c$ miles.

A walking at the rate of p miles an hour will travel x miles in $\frac{x}{p}$ hours; and B will travel $x-c$ miles in $\frac{x-c}{q}$ hours; these two times being equal, we have

$$\frac{x}{p}=\frac{x-c}{q},$$

$$qx=px-pc;$$

whence
$$x=\frac{pc}{p-q}.$$

Therefore A has travelled $\frac{pc}{p-q}$ miles.

Example 5. A train travelled a certain distance at a uniform rate. Had the speed been 6 miles an hour more, the journey would have occupied 4 hours less; and had the speed been 6 miles an hour less, the journey would have occupied 6 hours more. Find the distance.

Let the speed of the train be x miles per hour, and let the time occupied be y hours; then the distance traversed will be represented by xy miles.

On the first supposition the speed per hour is $x+6$ miles, and the time taken is $y-4$ hours. In this case the distance traversed will be represented by $(x+6)(y-4)$ miles.

On the second supposition the distance traversed will be represented by $(x-6)(y+6)$ miles.

All these expressions for the distance must be equal;

$$\therefore \quad xy=(x+6)(y-4)=(x-6)(y+6).$$

From these equations we have

$$xy=xy+6y-4x-24,$$

or
$$6y-4x=24 \quad \ldots\ldots\ldots\ldots (1);$$

and
$$xy=xy-6y+6x-36,$$

or
$$6x-6y=36 \quad \ldots\ldots\ldots\ldots (2).$$

From (1) and (2) we obtain $x=30$, $y=24$.

Hence the distance is 720 miles.

EXAMPLES XXIV.

1. If the numerator of a fraction is increased by 5 it reduces to $\frac{2}{3}$, and if the denominator is increased by 9 it reduces to $\frac{1}{3}$: find the fraction.

2. Find a fraction such that it reduces to $\frac{3}{5}$ if 7 be subtracted from its denominator, and reduces to $\frac{3}{8}$ on subtracting 3 from its numerator.

3. If unity is taken from the denominator of a fraction it reduces to $\frac{1}{2}$; if 3 is added to the numerator it reduces to $\frac{4}{7}$: required the fraction.

4. Find a fraction which becomes $\frac{3}{4}$ on adding 5 to the numerator and subtracting 1 from the denominator, and reduces to $\frac{1}{3}$ on subtracting 4 from the numerator and adding 7 to the denominator.

5. If 9 is added to the numerator a certain fraction will be increased by $\frac{1}{3}$; if 6 is taken from the denominator the fraction reduces to $\frac{2}{3}$: required the fraction.

6. At what time between 9 and 10 o'clock are the hands of a watch together?

7. When are the hands of a clock 8 minutes apart between the hours of 5 and 6?

8. At what time between 10 and 11 o'clock is the hour-hand six minutes ahead of the minute-hand?

9. At what time between 1 and 2 o'clock are the hands of a watch in the same straight line?

10. When are the hands of a clock at right angles between the hours of 5 and 6?

11. At what times between 12 and 1 o'clock are the hands of a watch at right angles?

12. A person buys 20 yards of cloth and 25 yards of canvas for \$35. By selling the cloth at a gain of 15 per cent. and the canvas at a gain of 20 per cent. he clears \$5.75; find the price of each per yard.

13. A dealer spends \$1445 in buying horses at \$75 each and cows at \$20 each; through disease he loses 20 per cent. of the horses and 25 per cent. of the cows. By selling the animals at the price he gave for them he receives \$1140; find how many of each kind he bought.

14. The population of a certain district is 33000, of whom 835 can neither read nor write. These consist of 2 per cent. of all the males and 3 per cent. of all the females: find the number of males and females.

15. Two persons C and D start simultaneously from two places a miles apart, and walk to meet each other; if C walks p miles per hour, and D one mile per hour faster than C, how far will D have walked when they meet?

16. A can walk a miles per hour faster than B; supposing that he gives B a start of c miles, and that B walks n miles per hour, how far will A have walked when he overtakes B?

17. A, B, C start from the same place at the rates of a, $a+b$, $a+2b$ miles an hour respectively. B starts n hours after A, how long after B must C start in order that they may overtake A at the same instant, and how far will they then have walked?

18. Find the distance between two towns when by increasing the speed 7 miles per hour, a train can perform the journey in 1 hour less, and by reducing the speed 5 miles per hour can perform the journey in 1 hour more.

19. A person buys a certain quantity of land. If he had bought 7 acres more each acre would have cost \$4 less, and if each acre had cost \$18 more he would have obtained 15 acres less: how much did he pay for the land?

20. A can walk half a mile per hour faster than B, and three-quarters of a mile per hour faster than C. To walk a certain distance C takes three-quarters of an hour more than B, and two hours more than A: find their rates of walking per hour.

21. A man pays \$90 for coal; if each ton had cost 50 cents more he would have received 2 tons less, but if each ton had cost 75 cents less he would have received 4 tons more; how many tons did he buy?

22. A and B are playing for money; in the first game A loses one-half of his money, but in the second he wins one quarter of what B then has. When they cease playing, A has won \$10, and B has still \$25 more than A; with what amounts did they begin?

23. The area of three fields is 516 acres, and the area of the largest and smallest fields exceeds by 30 acres twice the area of the middle field. If the smallest field had been twice as large, and the other two fields half their actual size, the total area would have been 42 acres less than it is; find area of each of the fields.

24. A, B, C each spend the same amount in buying different qualities of cloth. B pays three-eighths of a dollar per yard less than A and obtains three-fourths of a yard more; C pays five-eighths of a dollar per yard more than A and obtains one yard less; how much does each spend?

25. B pays \$28 more rent for a field than A; he has three-fourths of an acre more and pays \$1.75 per acre more. C pays \$72.50 more than A; he has six and one-fourth acres more, but pays 25 cents per acre less; find the size of the fields.

MISCELLANEOUS EXAMPLES IV.

1. When $a=-3$, $b=5$, $c=-1$, $d=0$, find the value of

$$26c\sqrt[3]{a^3}-c^2d+5bc-4ac+d^2.$$

2. Solve the equations:

(1) $\frac{1}{3}x-\frac{3}{7}y=8-2x, \quad \frac{1}{2}y-3x=3-y;$

(2) $1=y+z=2(z+x)=3(x+y).$

3. Simplify

(1) $\dfrac{a-x}{a+x}-\dfrac{4x^2}{a^2-x^2}+\dfrac{a-3x}{x-a};$

(2) $\dfrac{b^2-3b}{b^2-2b+4}\times\dfrac{b^2+b-30}{b^2+3b-18}\div\dfrac{b^3-3b^2-10b}{b^3+8}.$

4. Find the square root of

$$9-36x+60x^2-\frac{160}{3}x^3+\frac{80}{3}x^4-\frac{64}{9}x^5+\frac{64}{81}x^6.$$

5. In a base-ball match the errors in the first four innings are one-fourth of the runs, and in the last five innings the errors are one-third of the runs. The score is 16, and the errors number 5; find the score in the first four innings.

6. Find the value of $\dfrac{a^2-x^2}{\dfrac{1}{a^2}-\dfrac{2}{ax}+\dfrac{1}{x^2}}\times\dfrac{\dfrac{1}{a^2x^2}}{a+x}$

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7. Find the value of

$$\frac{1}{3}(a+2)-3\left(1-\frac{1}{6}b\right)-\frac{2}{3}\left(2a-3b+\frac{3}{2}\right)+\frac{3}{2}b-4\left(\frac{1}{2}a-\frac{1}{3}\right).$$

8. Resolve into factors

(1) $3a^2-20a-7;$ (2) $a^4b^2-b^4a^2.$

9. Reduce to lowest terms $\dfrac{4x^3+7x^2-x+2}{4x^3+5x^2-7x-2}.$

10. Solve the equations:

(1) $x + 3 - \dfrac{x-12}{3} = \dfrac{x-4}{2} + \dfrac{x-8}{4}$;

(2) $x + y - z = 0, \quad x - y + z = 4, \quad 5x + y + z = 20$;

(3) $\dfrac{ax+b}{c} + \dfrac{dx+e}{f} = 1.$

11. Simplify $\dfrac{x+3}{x^2-5x+6} - \dfrac{x+2}{x^2-9x+14} + \dfrac{4}{x^2-10x+21}.$

12. A purse of sovereigns is divided amongst three persons, the first receiving half of them and one more, the second half of the remainder and one more, and the third six. Find the number of sovereigns the purse contained.

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13. If $h = -1$, $k = 2$, $l = 0$, $m = 1$, $n = -3$, find the value of

$$\frac{h^3(m-l) - \sqrt{3hn} + hk}{m(l-h) - 2hm^2 + \sqrt[3]{4hk}}.$$

14. Find the L.C.M. of

$15(p^3+q^3)$, $5(p^2-pq+q^2)$, $4(p^2+pq+q^2)$, $6(p^2-q^2)$.

15. Find the square root of

(1) $\dfrac{4x^2}{9} + \dfrac{9}{4x^2} - 2$;

(2) $1 - 6a + 5a^2 + 12a^3 + 4a^4$.

16. Simplify $\dfrac{20x^2+27x+9}{15x^2+19x+6} + \dfrac{20x^2+27x+9}{12x^2+17x+6}.$

17. Solve the equations:

(1) $\dfrac{a(x-b)}{a-b} + \dfrac{b(x-a)}{b-a} = 1$;

(2) $\dfrac{9}{x-4} + \dfrac{3}{x-8} = \dfrac{4}{x-9} + \dfrac{8}{x-3}.$

18. A sum of money is to be divided among a number of persons; if \$8 is given to each there will be \$3 short, and if \$7.50 is given to each there will be \$2 over: find the number of persons.

19. Resolve into factors:

(1) $2x^2 - 3ab + (a - 6b)x$; (2) $4x^2 - 4xy - 15y^2$.

20. In the expression $x^3 - 2x^2 + 3x - 4$, substitute $a - 2$ for x, and arrange the result according to the descending powers of a.

21. Simplify

(1) $\dfrac{x}{1 - \dfrac{1}{1+x}}$; (2) $\dfrac{x^2}{a + \dfrac{x^2}{a + \dfrac{x^2}{a}}}$.

22. Find the H.C.F. of

$3x^3 - 11x^2 + x + 15$ and $5x^4 - 7x^3 - 20x^2 - 11x - 3$.

23. Express in the simplest form

(1) $\dfrac{\dfrac{x}{y} - \dfrac{y}{x}}{\dfrac{x+y}{y} - \dfrac{y+x}{x}}$; (2) $\left(\dfrac{x^3-1}{x-1} + \dfrac{x^3+1}{x+1}\right) \div \left(\dfrac{1}{x-1} + \dfrac{1}{x+1}\right)$.

24. A person possesses \$5000 stock, some at 3 per cent., four times as much at $3\frac{1}{2}$ per cent., and the rest at 4 per cent.: find the amount of each kind of stock when his income is \$176.

25. Simplify the expression

$$-3[(a+b) - \{(2a-3b) - (5a+7b-16c) - (-13a+2b-3c-5d)\}],$$

and find its value when $a = 1$, $b = 2$, $c = 3$, $d = 4$.

26. Solve the following equations:

(1) $11y - x = 10$, $11x - 101y = 110$;

(2) $x + y - z = 3$, $x + z - y = 5$, $y + z - x = 7$

27. Express the following fractions in their simplest form:

(1) $\dfrac{32x^3 - 2x + 12}{12x^5 - x^4 + 4x^2}$; (2) $\dfrac{1}{x + \dfrac{1}{1 + \dfrac{x+3}{2-x}}}$.

28. What value of a will make the product of $3 - 8a$ and $3a + 4$ equal to the product of $6a + 11$ and $3 - 4a$?

29. Find the L.C.M. of x^3-x^2-3x-9 and $x^3-2x^2-5x-12$.

30. A certain number of two digits is equal to seven times the sum of its digits: if the digit in the units' place be decreased by two and that in the tens' place by one, and if the number thus formed be divided by the sum of its digits, the quotient is 10. Find the number.

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31. Find the value of

$$\frac{6x^2-5xy-6y^2}{2x^2+xy-y^2}\times\frac{3x^2-xy-4y^2}{2x^2-5xy+3y^2}\div\frac{9x^2-6xy-8y^2}{2x^2-3xy+y^2}.$$

32. Resolve each of the following expressions into four factors:

(1) $4a^4-17a^2b^2+4b^4$; (2) x^8-256y^8.

33. Find the expression of highest dimensions which will divide $24a^4b-2a^3b^2-9ab^4$ and $18a^6+a^4b^2-6a^3b^3$ without remainder.

34. Find the square root of

(1) $x(x+1)(x+2)(x+3)+1$;

(2) $(2a^2+13a+15)(a^2+4a-5)(2a^2+a-3)$.

35. Simplify

$$x-\frac{2x-6}{x^2-6x+9}-3+\frac{x^2+3x-4}{x^2+x-12}.$$

36. A quantity of land, partly pasture and partly arable, is sold at the rate of \$60 per acre for the pasture and \$40 per acre for the arable, and the whole sum obtained is \$10000. If the average price per acre were \$50 the sum obtained would be 10 per cent. higher: find how much of the land is pasture, and how much arable.

CHAPTER XXV.

QUADRATIC EQUATIONS.

185. DEFINITION. An equation which contains the square of the unknown quantity, *but no higher power*, is called a **quadratic equation,** or an **equation of the second degree.**

If the equation contains both the square and the first power of the unknown it is called an *affected* quadratic; if it contains only the square of the unknown it is said to be a *pure* quadratic.

Thus $2x^2 - 5x = 3$ is an affected quadratic,

and $5x^2 = 20$ is a pure quadratic.

Pure Quadratic Equations.

186. A pure quadratic may be considered as a simple equation in which the *square* of the unknown quantity is to be found.

Example. Solve $\dfrac{9}{x^2-27} = \dfrac{25}{x^2-11}$.

Multiplying across, $9x^2 - 99 = 25x^2 - 675$;

$$\therefore \quad 16x^2 = 576;$$

$$\therefore \quad x^2 = 36;$$

and taking the square root of these equals, we have

$$x = \pm 6.$$

[In regard to the double sign see Art. 119.]

187. In extracting the square root of the two sides of the equation $x^2 = 36$, it might seem that we ought to prefix the double sign to the quantities on both sides, and write $\pm x = \pm 6$. But an examination of the various cases shows this to be unnecessary. For $\pm x = \pm 6$ gives the four cases:

$$+x = +6, \quad +x = -6, \quad -x = +6, \quad -x = -6,$$

and these are all included in the two already given, namely $x = +6$, $x = -6$. Hence, when we extract the square root of the two sides of an equation, it is sufficient to put the double sign before the square root of *one* side.

Affected Quadratic Equations.

188. The equation $x^2=36$ is an instance of the simplest form of quadratic equations. The equation $(x-3)^2=25$ may be solved in a similar way; for taking the square root of both sides, we have two *simple* equations,

$$x-3=\pm 5.$$

Taking the upper sign, $x-3=+5$, whence $x=8$;
taking the lower sign, $x-3=-5$, whence $x=-2$.
$\therefore$ the solution is $x=8$, or -2.

Now the given equation $(x-3)^2=25$
may be written $x^2-6x+(3)^2=25$,
or $x^2-6x=16$.
Hence, by retracing our steps, we learn that the equation

$$x^2-6x=16$$

can be solved by first adding $(3)^2$ or 9 to each side, and then extracting the square root; and the reason why we add 9 to each side is that this quantity added to the left side makes it a *perfect square*.

Now whatever the quantity a may be,

$$x^2+2ax+a^2=(x+a)^2,$$

and $$x^2-2ax+a^2=(x-a)^2;$$

so that if a trinomial is a perfect square, and *its highest power*, x^2, *has unity for its coefficient*, we must always have the term without x equal to the square of half the coefficient of x. If, therefore, the terms in x^2 and x are given, **the square may be completed by adding the square of half the coefficient of x.**

Example. Solve $x^2+14x=32$.

The square of half 14 is $(7)^2$.

$$\therefore \quad x^2+14x+(7)^2=32+49;$$

that is, $$(x+7)^2=81;$$

$$\therefore \quad x+7=\pm 9;$$

$$\therefore \quad x=-7+9, \text{ or } -7-9;$$

$$\therefore \quad x=2, \text{ or } -16.$$

189. When an expression is a perfect square, the *square terms* are always *positive*. **Hence, before completing the square the coefficient of x^2 should be made equal to $+1$.**

Example 1. Solve $7x = x^2 - 8$.

Transpose *so as to have the terms involving x on one side, and the square term positive.*

Thus $$x^2 - 7x = 8.$$

Completing the square, $$x^2 - 7x + \left(\frac{7}{2}\right)^2 = 8 + \frac{49}{4};$$

that is, $$\left(x - \frac{7}{2}\right)^2 = \frac{81}{4};$$

$$\therefore\ x - \frac{7}{2} = \pm\frac{9}{2};$$

$$\therefore\ x = \frac{7}{2} \pm \frac{9}{2};$$

$$\therefore\ x = 8, \text{ or } -1.$$

Example 2. Solve $4 - \dfrac{8}{3x+1} = \dfrac{3x^2+5}{3x+1}$.

Clearing of fractions, $12x + 4 - 8 = 3x^2 + 5$;
bringing the terms involving x to one side, we obtain

$$3x^2 - 12x = -9.$$

Divide throughout by 3; then

$$x^2 - 4x = -3;$$

$$\therefore\ x^2 - 4x + (2)^2 = 4 - 3;$$

that is, $$(x-2)^2 = 1;$$

$$\therefore\ x - 2 = \pm 1;$$

$$\therefore\ x = 3, \text{ or } 1.$$

EXAMPLES XXV. a.

Solve the equations:

1. $7(x^2 - 7) = 6x^2$.
2. $(x+8)(x-8) = 17$.
3. $(7+x)(7-x) = 24$.
4. $\dfrac{x^2+8}{x^2+20} = \dfrac{1}{2}$.
5. $\dfrac{11}{3-x} = 4(x+3)$.
6. $\dfrac{x(3x+5)+21}{(3x-2)(2x+3)} = 1$.
7. $x^2 + 2x = 8$.
8. $x^2 + 6x = 40$.
9. $x^2 + 35 = 12x$.
10. $x^2 + x = 6$.
11. $x^2 - 156 = x$.
12. $11x + 12 = x^2$.
13. $x^2 + 4x = 32$.
14. $9x + 36 = x^2$.
15. $x^2 + 15x - 34 = 0$.

Solve the equations:

16. $\frac{1}{3}(2x^2+7)-(6-x^2)=\frac{3}{7}(x^2+3)$.

17. $\frac{x+5}{x-2}=\frac{x+37}{2x-1}$.

18. $\frac{x+3}{x+2}+\frac{x-2}{x-3}=\frac{5}{(x+2)(x-3)}$.

19. $\frac{x^2-4x+115}{2x+2}=3x-5$.

20. $\frac{3x+1}{2}-\frac{4}{x-1}=x+2$.

190. We have shown that the square may readily be completed when the coefficient of x^2 is unity. All cases may be reduced to this by dividing the equation throughout by the coefficient of x^2.

Example 1. Solve $32-3x^2=10x$.

Transposing, $$3x^2+10x=32.$$

Divide throughout by 3, so as to make the coefficient of x^2 unity.

Thus $$x^2+\frac{10}{3}x=\frac{32}{3}.$$

Completing the square, $$x^2+\frac{10}{3}x+\left(\frac{5}{3}\right)^2=\frac{32}{3}+\frac{25}{9};$$

that is, $$\left(x+\frac{5}{3}\right)^2=\frac{121}{9};$$

$$\therefore\quad x+\frac{5}{3}=\pm\frac{11}{3};$$

$$\therefore\quad x=-\frac{5}{3}\pm\frac{11}{3}=2, \text{ or } -5\tfrac{1}{3}.$$

Example 2. Solve $5x^2+11x=12$.

Dividing by 5, $$x^2+\frac{11}{5}x=\frac{12}{5}.$$

Completing the square, $$x^2+\frac{11}{5}x+\left(\frac{11}{10}\right)^2=\frac{12}{5}+\frac{121}{100};$$

that is, $$\left(x+\frac{11}{10}\right)^2=\frac{361}{100};$$

$$\therefore\quad x+\frac{11}{10}=\pm\frac{19}{10};$$

$$\therefore\quad x=-\frac{11}{10}\pm\frac{19}{10}=\frac{4}{5}, \text{ or } -3.$$

191. We see then that the following steps are required for solving an affected quadratic equation:

(1) *If necessary, simplify the equation so that the terms in* x² *and* x *are on one side of the equation, and the term without* x *on the other.*

(2) *Make the coefficient of* x² *unity and positive by dividing throughout by the coefficient of* x².

(3) *Add to each side of the equation the square of half the coefficient of* x.

(4) *Take the square root of each side.*

(5) *Solve the resulting simple equations.*

192. When the coefficients are literal the same method may be used.

Example. Solve $7(x+2a)^2+3a^2=5a(7x+23a)$.

Simplifying, $7x^2+28ax+28a^2+3a^2=35ax+115a^2$;

that is, $7x^2-7ax=84a^2$,

or $x^2-ax=12a^2$.

Completing the square, $x^2-ax+\left(\dfrac{a}{2}\right)^2=12a^2+\dfrac{a^2}{4}$;

that is, $\left(x-\dfrac{a}{2}\right)^2=\dfrac{49a^2}{4}$;

$$\therefore \quad x-\frac{a}{2}=\pm\frac{7a}{2};$$

$$\therefore \quad x=4a, \text{ or } -3a.$$

193. In all the instances considered hitherto the quadratic equations have had two roots. Sometimes, however, there is only one solution. Thus if $x^2-2x+1=0$, then $(x-1)^2=0$, whence $x=1$ is the only solution. Nevertheless, in this and similar cases we find it convenient to say that the quadratic has *two equal roots.*

EXAMPLES XXV. b.

Solve the equations:

1. $3x^2+2x=21$. 2. $5x^2=8x+21$. 3. $6x^2-x-1=0$.

4. $3-11x=4x^2$. 5. $21x^2=2x+3$. 6. $10+23x+12x^2=0$.

7. $15x^2-6x=9$. 8. $4x^2-17x=15$. 9. $8x^2-19x-15=0$.

Solve the equations:

10. $10x^2+3x=1$. **11.** $12x^2+7x=12$. **12.** $20x^2-x-1=0$.

13. $x^2+2ax=15a^2$. **14.** $2x^2-8a^2=15ax$. **15.** $3x^2=k(2k-5x)$.

16. $11bx+20b^2=3x^2$. **17.** $9x^2-143c^2=6cx$. **18.** $2a^2x^2=ax+1$.

19. $(x-3)(x-2)=2(x^2-4)$. **20.** $5(x+1)(3x+5)=3(3x^2+11x+10)$.

21. $3x^2+13+(x-1)(2x+1)=2x(2x+3)$.

22. $\dfrac{7x-3}{x+1}=\dfrac{3x}{2}$. **23.** $\dfrac{2}{3x}=\dfrac{x-1}{2x-1}$. **24.** $\dfrac{3x-1}{x+9}=\dfrac{2x-9}{x-4}$.

25. $\dfrac{6x-5}{x+5}=\dfrac{x}{3}-5$. **26.** $\dfrac{x-4}{7x+1}+\dfrac{1}{2}=\dfrac{11}{2(3+2x)}$.

27. $3(2x+3)^2+2(2x+3)(2-x)=(x-2)^2$.

28. $(3x-7)^2-(2x-3)^2=(x-4)(3x+1)$.

[For additional examples see *Elementary Algebra*.]

194. Solution by Formula. From the preceding examples it appears that after suitable reduction and transposition every quadratic equation can be written in the form

$$ax^2+bx+c=0,$$

where a, b, c may have any numerical values whatever. If therefore we can solve this quadratic we can solve any.

Transposing, $$ax^2+bx=-c;$$

dividing by a, $$x^2+\frac{b}{a}x=-\frac{c}{a}.$$

Complete the square by adding to each side $\left(\dfrac{b}{2a}\right)^2$; thus

$$x^2+\frac{b}{a}\quad+\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a};$$

that is, $$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2};$$

extracting the square root,

$$x+\frac{b}{2a}=\frac{\pm\sqrt{(b^2-4ac)}}{2a};$$

$$\therefore\quad x=\frac{b\pm\sqrt{(b^2-4ac)}}{2a}.$$

195. In the result $x = \dfrac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$,

it must be remembered that the expression $\sqrt{(b^2 - 4ac)}$ is the square root of the compound quantity $b^2 - 4ac$, *taken as a whole.* We cannot simplify the solution unless we know the numerical values of a, b, c. It may sometimes happen that these values do not make $b^2 - 4ac$ a perfect square. In such a case the exact numerical solution of the equation cannot be determined.

Example. Solve $5x^2 - 13x - 11 = 0$.

Here $a = 5$, $b = -13$, $c = -11$; therefore by the formula we have

$$x = \frac{-(-13) \pm \sqrt{(-13)^2 - 4 \,.\, 5(-11)}}{2 \,.\, 5}$$

$$= \frac{13 \pm \sqrt{169 + 220}}{10}$$

$$= \frac{13 \pm \sqrt{389}}{10}.$$

Since 389 has not an exact square root this result cannot be simplified; thus the two roots are

$$\frac{13 + \sqrt{389}}{10}, \quad \frac{13 - \sqrt{389}}{10}.$$

196. Solution by Factors. There is still one method of obtaining the solution of a quadratic which will sometimes be found shorter than either of the methods already given.

Consider the equation $x^2 + \frac{7}{3}x = 2$.

Clearing of fractions, $3x^2 + 7x - 6 = 0$ ....(1);
by resolving the left-hand side into factors we have

$$(3x - 2)(x + 3) = 0.$$

Now if *either* of the factors $3x - 2$, $x + 3$ be zero, their product is zero. Hence the quadratic equation is satisfied by either of the suppositions

$$3x - 2 = 0, \text{ or } x + 3 = 0.$$

Thus the roots are $\frac{2}{3}$, -3.

It appears from this that *when a quadratic equation has been simplified and brought to the form of equation* (1), its solution can always be readily obtained if the expression on the left-hand

side can be resolved into factors. Each of these factors equated to zero gives a simple equation, and a corresponding root of the quadratic.

Example 1. Solve $2x^2 - ax + 2bx = ab$.

Transposing, *so as to have all the terms on one side of the equation*, we have

$$2x^2 - ax + 2bx - ab = 0.$$

Now
$$2x^2 - ax + 2bx - ab = x(2x - a) + b(2x - a)$$
$$= (2x - a)(x + b).$$

Therefore
$$(2x - a)(x + b) = 0\ ;$$
whence
$$2x - a = 0, \text{ or } x + b = 0,$$
$$\therefore \quad x = \frac{a}{2}, \text{ or } -b.$$

Example 2. Solve $2(x^2 - 6) = 3(x - 4)$.

We have
$$2x^2 - 12 = 3x - 12\ ;$$
that is,
$$2x^2 = 3x \quad \text{......................... (1)}.$$
Transposing,
$$2x^2 - 3x = 0,$$
$$x(2x - 3) = 0.$$
$$\therefore \quad x = 0, \text{ or } 2x - 3 = 0.$$

Thus the roots are $0, \frac{3}{2}$.

Note. In equation (1) above we might have divided both sides by x and obtained the simple equation $2x = 3$, whence $x = \frac{3}{2}$, which is *one* of the solutions of the given equation. But the student must be particularly careful to notice that whenever an x is removed by division from every term of an equation it must not be neglected, since the equation is satisfied by $x = 0$, which is therefore one of the roots.

197. Formation of Equations with given roots. It is now easy to form an equation whose roots are known.

Example 1. Form the equation whose roots are 4 and -3.

Here
$$x = 4, \text{ or } \quad x = -3\ ;$$
$$\therefore \quad x - 4 = 0, \text{ or } x + 3 = 0\ ;$$
both of these statements are included in
$$(x - 4)(x + 3) = 0,$$
or
$$x^2 - x - 12 = 0,$$
which is the required equation.

Example 2. Form the equation whose roots are a and $-\frac{b}{3}$.

Here $x = a$, or $x = -\frac{b}{3}$;

$\therefore$ the equation is $(x-a)\left(x+\frac{b}{3}\right) = 0$:

that is, $(x-a)(3x+b) = 0$,

or $3x^2 - 3ax + bx - ab = 0$.

EXAMPLES XXV. c.

Solve by formula the equations:

1. $x^2+2x-3=0$.
2. $x^2-2x-1=0$.
3. $x^2-3x=5$.
4. $3x^2-2x=1$.
5. $2x^2-9x=4$.
6. $3x^2+7x=6$.
7. $4x^2-14=3x$.
8. $6x^2-3-7x=0$.
9. $12x^2+10=23x$.

Solve by resolution into factors:

10. $x^2-9x=90$.
11. $x^2-11x=152$.
12. $x^2-85=12x$.
13. $2x^2-3x=2$.
14. $3x^2+5x+2=0$.
15. $4x^2-14=x$.
16. $5x^2-11x+2=0$.
17. $x^2-a^2=0$.
18. $x^2-7ax=8a^2$.
19. $12x^2-23bx+10b^2=0$.
20. $3ax^2+2bx=7x$.
21. $24x^2+22cx=21c^2$.
22. $x^2-2x+4b=2bx$.

Solve the equations:

23. $2x(x+9)=(x+1)(5-x)$.
24. $(2x-1)^2-11=5x+(x-3)^2$.
25. $6(x-2)^2+13(1-x)(x-2)+6x^2=6(2x-1)$.
26. $\frac{3}{x-6}-\frac{4}{x-5}=1$.
27. $\frac{3}{2x-1}-\frac{2}{x+1}=\frac{2}{x}$.
28. $\frac{10}{x-4}-\frac{9}{x}=\frac{4}{x-5}$.
29. $\frac{x+3}{x-3}+\frac{4(x-6)}{x+6}=3$.
30. $3x=\frac{1}{x+1}+2$.
31. $\frac{x}{3}-\frac{6}{x}=\frac{x^2-6}{3(x+4)}$.
32. $\frac{23}{x+4}+\frac{3x}{11}=\frac{1}{3}(x+5)$.
33. $\frac{x+1}{x-3}-\frac{5}{x}=6$.

Solve the equations:

34. $\dfrac{x+2}{x-1} - \dfrac{4-x}{2x} = 2\tfrac{1}{3}$.

35. $\dfrac{21x^3-16}{3x^2-4} - 7x = 5$.

36. $\dfrac{5x+1}{x+8} + \dfrac{x-8}{3x-1} = \dfrac{17}{12}$.

37. $\dfrac{1}{2(x-1)} + \dfrac{3}{x^2-1} = \dfrac{1}{4}$.

38. $\dfrac{a^2(x-b)}{a-b} - x^2 = \dfrac{b^2(a-x)}{b-a}$.

39. $(p-q)x + \dfrac{2q}{x} = (p+q)$.

40. $\dfrac{b}{x-a} + \dfrac{a}{x-b} = 2$.

41. $(x-1)^2 = \left(\dfrac{b}{c} - \dfrac{c}{b}\right)^2 x$.

[For additional examples see *Elementary Algebra.*]

198. Simultaneous Quadratic Equations. If from either of two equations which involve x and y the value of one of the unknowns can be expressed in terms of the other, then by substitution in the second equation we obtain a quadratic which may be solved by any one of the methods explained in this chapter.

Example. Solve the simultaneous equations

$$5x+7y=1, \quad 4x^2+3xy-2y^2=10.$$

From the first equation, $x=\dfrac{1-7y}{5}$, and therefore by substitution in the second equation, we have

$$\frac{4(1-7y)^2}{25}+\frac{3y(1-7y)}{5}-2y^2=10;$$

whence $\quad 4-56y+196y^2+15y-105y^2-50y^2=250;$

that is, $\quad 41y^2-41y-246=0;$

$$\therefore \quad y^2-y-6=0;$$

$$\therefore \quad (y-3)(y+2)=0;$$

$$\therefore \quad y=3, \text{ or } -2.$$

From the first equation, we see that if $y=3$, then $x=-4$, and if $y=-2$, then $x=3$.

Homogeneous Equations of the Same Degree.

199. The most convenient method of solution is to substitute $y=mx$ in each of the given equations. By division we eliminate x and obtain a quadratic to determine the values of m.

Example. Solve the simultaneous equations

$$5x^2+3y^2=32, \quad x^2-xy+2y^2=16.$$

Put $y=mx$ and substitute in each equation. Thus

$$x^2(5+3m^2)=32 \quad \text{.........................(1)},$$

and $$x^2(1-m+2m^2)=16 \quad \text{.........................(2)}.$$

By division, $$\frac{5+3m^2}{1-m+2m^2}=\frac{32}{16}=2;$$

that is, $$m^2-2m-3=0;$$

$$\therefore \quad (m-3)(m+1)=0;$$

$$\therefore \quad m=3, \text{ or } -1.$$

(1) Take $m=3$ and substitute in either (1) or (2).

From (1), $32x^2=32$; whence $x=\pm 1$.

$$\therefore \quad y=mx=3x=\pm 3.$$

(2) Take $m=-1$ and substitute in (1). Thus

$8x^2=32$; whence $x=\pm 2$.

$$\therefore \quad y=mx=-x=\mp 2.$$

EXAMPLES XXV. d.

Solve the simultaneous equations :

1. $x+3y=9,$ $\quad xy=6.$

2. $3x-4y=2,$ $\quad xy=2.$

3. $2x+y=5,$ $\quad 5x^2-xy=2.$

4. $x-2y=3,$ $\quad x^2+4y^2=29.$

5. $3x+y=9,$ $\quad 3xy-y^2=9.$

6. $2x-5y=1,$ $\quad x^2-8y^2=1.$

7. $\frac{x}{2}-y=1,$ $\quad xy=24.$

8. $\frac{3}{x}-\frac{1}{y}=1,$ $\quad 10xy=1.$

9. $x-\frac{y}{2}=3,$ $\quad xy-y^2=4.$

10. $\frac{x}{2}-\frac{1}{y}=2,$ $\quad \frac{3}{x}+\frac{y}{2}=1.$

11. $\frac{x}{2}+\frac{y}{3}=3,$ $\quad \frac{8}{x}-\frac{3}{y}=1.$

12. $2x-\frac{9}{y}=1,$ $\quad 3y-\frac{2}{x}=8.$

13. $3x^2+7y^2=55,$ $\quad 2x^2+7xy=60.$

14. $16xy-3x^2=77,$ $\quad 7xy+3y^2=110.$

15. $2x^2+5y^2=143,$ $\quad 8xy+3y^2=195.$

16. $x^2+2xy+2y^2=17,$ $\quad 3x^2-9xy-y^2=119.$

17. $21x^2+3xy-y^2=371,$ $\quad 5x^2+3xy+5y^2=265.$

CHAPTER XXVI.

PROBLEMS LEADING TO QUADRATIC EQUATIONS.

200. WE shall now discuss some problems which give rise to quadratic equations.

Example 1. A train travels 300 miles at a uniform rate; if the speed had been 5 miles an hour more, the journey would have taken two hours less: find the rate of the train.

Suppose the train travels at the rate of x miles per hour, then the time occupied is $\frac{300}{x}$ hours.

On the other supposition the time is $\frac{300}{x+5}$ hours;

$$\therefore \quad \frac{300}{x+5}=\frac{300}{x}-2;$$

whence $$x^2+5x-750=0,$$

or $$(x+30)(x-25)=0,$$

$$\therefore \quad x=25, \text{ or } -30.$$

Hence the train travels 25 miles per hour, the negative value being inadmissible.

[For an explanation of the meaning of the negative value see *Elementary Algebra*.]

Example 2. A man buys a number of articles for \$2.40, and sells for \$2.52 all but two at 2 cents apiece more than they cost; how many did he buy?

Let x be the number of articles bought; then the cost price of each is $\frac{240}{x}$ cents, and the sale price is $\frac{252}{x-2}$ cents.

$$\therefore \quad \frac{252}{x-2}-\frac{240}{x}=2$$

that is, $$\frac{126}{x-2}-\frac{120}{x}=1.$$

After simplification, $6x+240=x^2-2x,$

or $x^2-8x-240=0;$

that is, $(x-20)(x+12)=0;$

$\therefore \quad x=20, \text{ or } -12.$

Thus the number required is 20.

Example 3. A cistern can be filled by two pipes in $33\frac{1}{3}$ minutes; if the larger pipe takes 15 minutes less than the smaller to fill the cistern, find in what time it will be filled by each pipe singly.

Suppose that the two pipes running singly would fill the cistern in x and $x-15$ minutes; then they will fill $\frac{1}{x}$ and $\frac{1}{x-15}$ of the cistern respectively in one minute, and therefore when running together they will fill $\left(\frac{1}{x}+\frac{1}{x-15}\right)$ of the cistern in one minute.

But they fill $\frac{1}{33\frac{1}{3}}$, or $\frac{3}{100}$ of the cistern in one minute.

Hence $$\frac{1}{x}+\frac{1}{x-15}=\frac{3}{100},$$

$$100(2x-15)=3x(x-15),$$

$$3x^2-245x+1500=0,$$

$$(x-75)(3x-20)=0;$$

$$\therefore \quad x=75, \text{ or } 6\tfrac{2}{3}.$$

Thus the smaller pipe takes 75 minutes, the larger 60 minutes.

The other solution $6\frac{2}{3}$ is inadmissible.

201. Sometimes it will be found convenient to use more than one unknown.

Example. Nine times the side of one square exceeds the perimeter of a second square by one foot, and six times the area of the second square exceeds twenty-nine times the area of the first by one square foot: find the length of a side of each square.

Let x feet and y feet represent the sides of the two squares; then the perimeter of the second square is $4y$ feet; thus

$$9x-4y=1.$$

The areas of the two squares are x^2 and y^2 square feet; thus

$$6y^2-29x^2=1.$$

From the first equation, $$y = \frac{9x-1}{4}.$$

By substitution in the second equation,

$$\frac{3(9x-1)^2}{8} - 29x^2 = 1;$$

that is, $$11x^2 - 54x - 5 = 0,$$

or $$(x-5)(11x+1) = 0;$$

whence $x = 5$, the negative value being inadmissible.

Also, $$y = \frac{9x-1}{4} = 11.$$

Thus the lengths are 5 ft. and 11 ft.

EXAMPLES XXVI.

1. Find a number which is less than its square by 72.

2. Divide 16 into two parts such that the sum of their squares is 130.

3. Find two numbers differing by 5 such that the sum of their squares is equal to 233.

4. Find a number which when increased by 13 is 68 times the reciprocal of the number.

5. Find two numbers differing by 7 such that their product is 330.

6. The breadth of a rectangle is 5 yards shorter than the length, and the area is 374 square yards: find the sides.

7. One side of a rectangle is 7 yards longer than the other, and its diagonal is 13 yards: find the area.

8. Find two consecutive numbers the difference of whose reciprocals is $\frac{1}{306}$.

9. Find two consecutive even numbers the difference of whose reciprocals is $\frac{1}{480}$.

10. The difference of the reciprocals of two consecutive odd numbers is $\frac{2}{675}$: find them.

11. A farmer bought a certain number of sheep for \$315; through disease he lost 10, but by selling the remainder at 75 cents each more than he gave for them, he gained \$75: how many did he buy?

12. By walking three-quarters of a mile more than his ordinary pace per hour, a man finds that he takes $1\frac{1}{2}$ hours less than usual to walk $29\frac{3}{4}$ miles: what is the ordinary rate?

13. A cistern can be filled by the larger of two pipes in 5 minutes less than by the smaller. When the taps are both running the cistern is filled in 6 minutes: find the time in which the cistern could be filled by each of the pipes.

14. A man buys a dozen eggs, and calculates that if they had been a cent per dozen cheaper he could have bought two more for twelve cents: what is the price per dozen?

15. The large wheel of a carriage is one foot more in circumference than the small wheel, and makes 48 revolutions less per mile: find the circumference of each wheel.

16. A boy was sent out to buy 12 cents' worth of apples. He ate two, and his master had in consequence to pay at the rate of a cent per dozen more than the market price. How many apples did the boy buy?

17. A lawn 45 feet long and 40 broad has a path of uniform width round it; if the area of the path is 50 square yards, find its width.

18. By selling one more apple for a cent than she formerly did, a woman finds that she gets a cent less per dozen: how much does she now get per dozen?

19. Four times the side of one square is less than the perimeter of a second square by 12 feet, and eleven times the area of the first is less than five times the area of the second by 9 square feet: find the length of a side of each square.

20. Find a number of two digits such that if it be divided by the product of its digits the quotient is 7, and if 27 be subtracted from the number the order of the digits is reversed. [Art. 111.]

21. A person buys some $5\frac{1}{2}$ per cent. stock; if the price had been \$5 less he would have received one per cent. more interest on his money: at what price did he buy the stock?

22. The area of each of two rectangles is 1008 square feet; the length of one is 8 feet more than that of the other, and the difference of their breadths is 3 feet: find their sides.

23. There are three numbers of which the second is greater than the first by 6 and less than the third by 9. If the product of all three is 280 times the greatest, find the numbers.

24. Find four consecutive integers such that the product of the two greatest is represented by a number which has the two least for its digits.

25. Two trains A and B start simultaneously from two stations P and Q which are 260 miles apart. A reaches Q in $3\frac{3}{7}$ hours, and B reaches P in $4\frac{2}{3}$ hours after they meet: find the rate of each train.

MISCELLANEOUS EXAMPLES V.

[*The following examples are arranged progressively: 1–24 may be taken after* CHAP. XIII.; 25–36 *after* CHAP. XVII.; 37–48 *after* CHAP. XX.; 49–60 *after* CHAP. XXII.; 61–72 *after* CHAP. XXIV. *The remaining examples are quite general and cover the contents of the whole book.*]

1. If $x = 3$, $y = -2$, $z = 0$, find the value of

$$\frac{3x^2+5yz+4y^3}{x-4y+z}.$$

2. Divide $3p^5+16p^4-33p^3+14p^2$ by p^2+7p.

3. Find the sum of $a-2(b-3c)$, $3\{a-2(b+c)\}$, $2\{b-2(a-2b)\}$.

4. Simplify by removing brackets $7[3a-4\{a-b+3(a+b)\}]$.

5. Solve the equations:

(1) $\dfrac{x+4}{7}+\dfrac{x-4}{3}=4$;

(2) $\dfrac{2}{5}(x-4)+\dfrac{2+x}{2}+\dfrac{2x-1}{7}=7-\dfrac{23-2x}{5}$.

6. A is three times as old as B; two years ago he was five times as old as B was four years ago: what is A's age?

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7. Find the product of $2a-3b-(a-2b-c)$ and $b-2c-(a-c)$.

8. If $a=1$, $b=0$, $c=-1$, $d=2$, $e=-2$, find the value of

$$a^3-b^3+c^3+d^3-e^3+a^2+b^2+c^2+d^2-e^2.$$

9. Remove brackets from the expressions:

(1) $a-[5b-\{a-(3c-3b)+2c-(a-2b-c)\}]$;

(2) $2[a-3\{b-4(c-d)\}]-[a-4\{b-6(c-d)\}]$.

10. If the price of 5 acres of land is $\$a$, what is the price of x acres? and how many acres can be bought for $\$b$?

11. Divide a^4-4 by a^2-2a+2.

12. There are 150 coins in a bag which are either half-dollars or quarters. If the value of the coins is \$58.50, find the number of each kind.

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13. Add together $a-\{b+c-(a+b)\}+c$, $2(3a+2b)-4(b+2a)-c$, and $3(2b-a)-2(3b-a)+c$.

14. Find what value of x will make the product of $x+3$ and $2x+3$ exceed the product of $x+1$ and $2x+1$ by 14.

15. Divide $b^3+8-125c^3+30bc$ by $b-5c+2$.

16. Simplify

$$(1)\quad \frac{13ab^2c^3}{29c^4d}\times\frac{87c^2d^2}{2a^2b^4}\div\frac{26bcd}{4ab^3};$$

$$(2)\quad \frac{2ax}{b^2}-\left(\frac{a^2xc^2}{2ab^2c^2}+\frac{a^3bx}{6a^2b^3}+\frac{a^4b^2x^4}{2a^3b^4x^3}\right).$$

17. How old will a man be in m years who n years ago was p times as old as his son then aged x years?

18. I bought a certain number of pears at three for a cent, and two-thirds of that number at four for a cent; by selling them at twenty-five for 12 cents, I gained 18 cents. How many pears did I buy?

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19. Solve the equations:

$$(1)\quad \frac{17-3x}{5}-\frac{4x+2}{3}=5-6x+\frac{7x+14}{3};$$

$$(2)\quad \frac{x+y}{2}+\frac{3x-5y}{4}=2,\quad \frac{x}{14}+\frac{y}{18}=1.$$

20. Divide $a^2x^8+(2ac-b^2)x^4+c^2$ by ax^4+c-bx^2.

21. If a horses are worth b cows, and c cows are worth d sheep, find the value of a horse when a sheep is worth \$2.

22. Find the highest common factor of $3a^2b^3c$, $12a^3bc^2$, $15a^2b^5$; and the lowest common multiple of $4ab^2c^3$, $12a^3b$, $18ac^2$.

Also find the value of $\dfrac{a^2c}{2ab^2c}-\dfrac{a}{3b^2}+\dfrac{ad}{b^2d}-\dfrac{a^3x^2}{6a^2b^2x^2}$.

23. A gentleman divided \$49 amongst 150 children. Each girl had 50 cents, and each boy 25 cents. How many boys were there?

24. If $V=5a+4b-6c$, $X=-3a-9b+7c$, $Y=20a+7b-5c$, $Z=13a-5b+9c$, calculate the value of $V-(X+Y)+Z$.

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25. Solve the equations :

(1) $\frac{3(6-5x)}{5}+\frac{63x}{50}=\frac{3x}{2}-\frac{36}{125}$;

(2) $\frac{1}{12}(x+y)=x+1, \qquad \frac{1}{6}(y-x)=2x-1.$

26. Find the factors of

(1) $a^2-a-182$; (2) $8x^2+13x-6$.

27. If $x=4$, $y=5$, and $z=3$, find the value of

$$\sqrt[3]{\{5(y^2-z^2)-x^2\}}+\sqrt[4]{3\{x(x^2-z^2)-1\}}.$$

28. The product of two expressions is $(x+2y)^3+(3x+z)^3$, and one of them is $4x+2y+z$; find the other.

29. When A and B sit down to play, B has two-thirds as much money as A; after a time A wins \$15, and then he has twice as much money as B. How much had each at first?

30. Find the square root of $16a^6+4a+4-16a^3+a^2-8a^4$.

31. Find the value of

$$24\left\{x-\frac{1}{2}(x-3)\right\}\left\{x-\frac{2}{3}(x+2)\right\}\left\{x-\frac{3}{4}\left(x-\frac{4}{3}\right)\right\},$$

and subtract the result from $(x+2)(x-3)(x+4)$.

32. Find the square root of

$$\frac{x^4}{4}-\frac{2x^3}{3}-\frac{11x^2}{36}+x+\frac{9}{16}.$$

33. If $2a=3b=c=4d=1$, find the value of

$$\frac{a^2c}{d}-\frac{b}{3c}+\sqrt{a^4d}-\sqrt[3]{a^4d}+\frac{d}{9a^2}.$$

34. Separate into their simplest factors :

(1) $x^2-xy-6y^2$; (2) $x^3-4xy^2-x^2y+4y^3$.

35. Solve the equations :

(1) $(x-1)(x-2)(x-6)=(x-3)^3$;

(2) $\frac{3}{2x}+\frac{2}{3y}=5, \qquad \frac{2}{x}+\frac{3}{y}=13.$

36. A farmer sells to one person 9 horses and 7 cows for \$375, and to another 6 horses and 13 cows at the same prices and for the same sum : what was the price of each?

37. When $a=3$, $b=2$, $c=-7$, find the value of

(1) $\frac{3b^2c}{a}+\frac{5bc^2}{a-c}-\frac{6a^3}{b+2c}$;

(2) $4c+\{c-(3c-2b)+2b\}$.

38. Solve the equations :

(1) $\frac{1}{x}+\frac{1}{2x}-\frac{1}{3x}=\frac{7}{3}$;

(2) $5x+3y=120$, $\quad 10x=9y+90$.

39. Find the highest common factor of $5x^3+2x^2-15x-6$ and $7x^3-4x^2-21x+12$.

40. A coach travels between two places in 5 hours ; if its speed were increased by 3 miles an hour, it would take $3\frac{1}{3}$ hours for the journey : what is the distance between the places ?

41. From $x(x+a-b)(x-a+b)$ take $(x-a)(x-b)(x+a+b)$.

42. Find the value of

$$\frac{2x^2+5x-3}{x^3-9x}\times\frac{3x^2-10x+3}{x^2+3x+2}\div\frac{6x^2-5x+1}{3x^3+7x^2+2x}.$$

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43. Divide $x^3+y^3+3xy-1$ by $x+y-1$, and extract the square root of $x^4-3x^3+\frac{11}{12}x^2+2x+\frac{4}{9}$.

44. A man can walk from A to B and back in a certain time at the rate of 4 miles an hour. If he walks at the rate of 3 miles an hour from A to B, and at the rate of 5 miles an hour from B to A, he requires 10 minutes longer for the double journey. What is the distance from A to B ?

45. Find the highest common factor of

$7x^4-10ax^3+3a^2x^2-4a^3x+4a^4$, $\quad 8x^4-13ax^3+5a^2x^2-3a^3x+3a^4$.

46. Solve the equations :

(1) $x-\left(3x-\frac{2x-5}{10}\right)=\frac{1}{6}(2x-57)-\frac{5}{3}$;

(2) $x-2y+z=0$, $\quad 9x-8y+3z=0$, $\quad 2x+3y+5z=36$.

47. Find the lowest common multiple of

$6x^2-x-1$, $\quad 3x^2+7x+2$, $\quad 2x^2+3x-2$.

48. The expression $ax+3b$ is equal to 30 when x is 3, and to 42 when x is 7: what is its value when x is 1; and for what value of x is it equal to zero?

49. Find the lowest common multiple of

$$4(a^2+ab), \quad 12(ab^2-b^3), \quad 18(a^2-b^2).$$

50. Extract the square root of

$$\frac{4x^2}{a^2}-\frac{12x}{a}+25-\frac{24a}{x}+\frac{16a^2}{x^2}.$$

51. Reduce to lowest terms

$$\frac{12x^4+4x^3-23x^2-9x-9}{8x^4-14x^2-9}.$$

52. Solve the equations:

(1) $x-\dfrac{3x-2y}{7}=y-\dfrac{4x-5y}{19}, \quad y-\dfrac{2x-7y}{6}=2(x-1)-\dfrac{3x-7}{8};$

(2) $2x-y+3z=1, \quad 4x+3y-2z=13, \quad 6x-4y+z=20.$

53. Simplify

(1) $\dfrac{a+b}{b}-\dfrac{2a}{a+b}+\dfrac{a^2b-a^3}{a^2b-b^3};$

(2) $\dfrac{1}{x^2+8x+15}-\dfrac{1}{x^2+11x+30}.$

54. The sum of the two digits of a number is 9; if the digits are reversed the new number is four-sevenths of what it was before. Find the number.

55. Solve the equations:

(1) $4x-\dfrac{1}{3}(5y-4)=1, \quad \dfrac{3y-2x}{4}+\dfrac{1}{3}x=\dfrac{1}{2};$

(2) $3x+4y-11=0, \quad 5y-6z=-8, \quad 7z-8x-13=0.$

56. Find the value of

$$\frac{2}{a+x}-\frac{1}{a-x}-\frac{3x}{x^2-a^2}-\frac{a}{(a+x)^2}.$$

57. Resolve into factors:

(1) $x^6-2x^4+x^2$; (2) $a^6+a^4-a^2-1$.

58. Two persons started at the same time to go from A to B. One rode at the rate of $7\frac{1}{2}$ miles per hour and arrived half an hour later than the other who travelled by train at the rate of 30 miles per hour. What is the distance between A and B?

59. Find the square root of

$$\frac{4x^2}{9y^2}-\frac{x}{z}-\frac{16x^2}{15yz}+\frac{9y^2}{16z^2}+\frac{6xy}{5z^2}+\frac{16x^2}{25z^2}.$$

60. Find the factor of highest dimensions which will exactly divide each of the expressions

$$2c^4+c^3d-c^2d^2-7cd^3-4d^4, \quad 3c^4+c^3d-2c^2d^2-9cd^3-5d^4.$$

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61. Simplify (1) $\dfrac{p+2}{2}-\dfrac{p}{p+2}-\dfrac{p^3-2p^2}{2p^2-8}$;

(2) $\left(1+\dfrac{3x}{a-x}\right)\left(\dfrac{a-x}{a+2x}\right)^2$.

62. Find the highest common factor of

$$6x^4-2x^3+9x^2+9x-4 \quad \text{and} \quad 9x^4+80x^2-9.$$

What value of x will make both these expressions vanish?

63. Solve the equations:

(1) $\dfrac{x+2}{x-3}+\dfrac{x-2}{x-6}=2$;

(2) $\dfrac{2x}{x-1}+\dfrac{3x-1}{x+2}-\dfrac{5x-15}{x-2}=0$.

64. Simplify $\dfrac{6x^2-5xy-6y^2}{14x^2-23xy+3y^2}-\dfrac{15x^2+8xy-12y^2}{35x^2+47xy+6y^2}$.

65. Find the value of

$$\frac{x-2a}{x+2b}-\frac{x+2a}{x-2b}-\frac{16ab}{4b^2-x^2}, \text{ when } x=\frac{4ab}{a+b}.$$

66. An egg-dealer bought a certain number of eggs at 16 cents per score, and five times the number at 75 cents per hundred; he sold the whole at 10 cents per dozen, gaining \$3.24 by the transaction. How many eggs did he buy?

67. If $a=-1$, $b=-2$, $c=-3$, $d=-4$, find the value of

$$\frac{2a^3+ab^2-3abc}{a^2-b^2-abc}-\frac{a-b+c-d}{bc-2ad}-\frac{a}{b}+\frac{2c}{ad}.$$

68. Solve the simultaneous equations:

$$\left.\begin{aligned}\frac{x-2}{2}-\frac{x+y}{14}&=\frac{x-y-1}{8}-\frac{y+12}{4},\\ \frac{x+7}{3}+\frac{y-5}{10}&=1-x-\frac{5(y+1)}{7}.\end{aligned}\right\}$$

69. Simplify the fractions:

(1) $$\frac{x-y}{x-z}+\frac{x-z}{x-y}-\frac{(y-z)^2}{(z-x)(y-x)};$$

(2) $$\frac{\dfrac{a^3}{b^3}-\dfrac{b^3}{a^3}}{\left(\dfrac{a}{b}-\dfrac{b}{a}\right)\left(\dfrac{a}{b}+\dfrac{b}{a}-1\right)}\div\frac{\dfrac{1}{b}-\dfrac{1}{a}}{\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{ab}}.$$

70. From a certain sum of money one-third part was taken and \$50 put in its stead. From the sum thus increased one-fourth part was taken and \$70 put in its stead. If the amount was now \$120, find the original sum.

71. Find the lowest common multiple of

$$(a^4-a^2c^2)^2, \quad 4a^6-8a^4c^2+4a^2c^4, \quad a^4+3a^3c+3a^2c^2+ac^3.$$

72. Solve the equations:

(1) $$\cdot15x+\frac{\cdot135x-\cdot225}{\cdot03}=\frac{\cdot36}{\cdot2}-\frac{\cdot09x-\cdot18}{\cdot9};$$

(2) $$\frac{10x+4}{21}+\frac{7-2x^2}{14(x-1)}=\frac{11-5x}{15}+\frac{4x-3\frac{2}{5}}{6}.$$

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73. Simplify

$$\frac{x^2-4x-21}{x+17}\times\frac{x^3+6x^2-247x}{x^2-x-12}\div\frac{x^2-20x+91}{x-4}.$$

74. What must be the value of x in order that

$$\frac{(a+2x)^2}{a^2+70ax+3x^2}$$

may be equal to $1\frac{1}{3}$ when a is equal to 67?

75. Find the highest common factor of $16x^4+36x^2+81$ and $8x^3+27$; and find the lowest common multiple of $8x^3+27$, $16x^4+36x^2+81$, and $6x^2-5x-6$.

76. Solve the equations:

(1) $a(x-a)-b(x-b)=(a+b)(x-a-b)$.

(2) $(a+b)x-ay=a^2, \quad (a^2+b^2)x-aby=a^3$.

77. A farmer bought a certain number of sheep for \$30. He sold all but five of them for \$27, and made a profit of 20 per cent. on those he sold: find how many he bought.

78. Find the value of

$$(1)\ \frac{\dfrac{x-y}{x+y}-\dfrac{x+y}{x-y}}{\dfrac{x^2-y^2}{x^2+y^2}-\dfrac{x^2+y^2}{x^2-y^2}}; \qquad (2)\ \frac{\dfrac{2a-3b}{2a-6b}-\dfrac{3b}{2a}}{\dfrac{2a-3b}{2a}+\dfrac{3b}{2a-6b}}.$$

---

79. Find the H.C.F. of $21x^3-26x^2+8x$ and $6a^2x^3-a^2x^2-2a^2x$. Also the L.C.M. of x^3-x, $ax^2+2ax-3a$, x^3-7x^2+6x.

80. Simplify the expression

$$(a+b+c)(a-b+c)-\{(a+c)^2-b^2-(a^2+b^2+c^2)\}.$$

81. Solve the equations:

$$(1)\ \frac{7+x}{5}-\frac{2x-y}{4}=3y-5, \qquad \frac{5y-7}{2}+\frac{4x-3}{6}=18-5x;$$

$$(2)\ \frac{4}{3x-2}-\frac{6}{x+1}=\frac{5}{2x+3}.$$

82. Extract the square root of

$$\frac{x^4}{y^4}+\frac{y^4}{x^4}-2\left(\frac{x^3}{y^3}+\frac{y^3}{x^3}\right)+3\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)-4\left(\frac{x}{y}+\frac{y}{x}\right)+5.$$

83. Find the value of

$$\frac{4a+6b}{a+b}+\frac{6a-4b}{a-b}-\frac{4a^2+6b^2}{a^2-b^2}+\frac{4b^2-6a^2}{a^2+b^2}+\frac{20b^4}{a^4-b^4}.$$

84. A bag contains 180 gold and silver coins of the value altogether of \$144. Each gold coin is worth as many cents as there are silver coins, and each silver coin as many cents as there are gold coins. How many coins are there of each kind?

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85. Solve the equations:

(1) $\frac{5}{x+10}+\frac{8}{x+4}=\frac{13}{x+7}$;

(2) $\sqrt{2x+6}-\sqrt{x-1}=2$.

86. Find the factors of

(1) $20a^2+21ab-27b^2$; (2) $x^3-3x^2-9x+27$.

87. If the length of a field were diminished and its breadth increased by 12 yards, it would be square. If its length were increased and its breadth diminished by 12 yards, its area would be 15049 square yards. Find the area of the field.

88. Simplify the expression

$$\frac{\{(a+b)(a+b+c)+c^2\}\{(a+b)^2-c^2\}}{\{(a+b)^3-c^3\}\{a+b+c\}}.$$

89. Find the square root of

$$(2x+1)(2x+3)(2x+5)(2x+7)+16.$$

90. Simplify

$$\frac{10a^2}{(1+a^2)(1-4a^2)}-\frac{1}{1-2a}+\frac{2}{1+a^2}.$$

---

91. Solve the equations:

(1) $\frac{x+6}{y}=\frac{3}{4}, \quad \frac{x}{y-2}=\frac{1}{2}$;

(2) $1{\cdot}2x-\frac{{\cdot}18x-{\cdot}05}{{\cdot}5}={\cdot}4x+8{\cdot}9$.

92. Resolve into factors:

(1) $6x^2+5x-6$. (2) $9x^4-82x^2y^2+9y^4$.

93. Reduce $\frac{x^4-15x^2+28x-12}{2x^3-15x+14}$

to its lowest terms.

94. Simplify the fraction

$$\frac{(a+b)^2}{(x-a)(x+a+b)}-\frac{a+2b+x}{2(x-a)}+\frac{(a+b)x}{x^2+bx-a^2-ab}+\frac{1}{2}.$$

95. A person being asked his age replied, "Ten years ago I was five times as old as my son, but twenty years hence he will be half my age." What is his age?

96. Find the value of

$$\left\{\frac{2a}{(a-x)(a+x)}+\frac{1}{x-a}-\frac{1}{x+a}\right\}\times\frac{x^2}{x-a+\dfrac{a^2}{x}}.$$

---

97. When $a=4$, $b=-2$, $c=\frac{3}{2}$, $d=-1$, find the value of

$$a^3-b^3-(a-b)^3-11(3b+2c)\left(2c^2-\frac{d^2}{2}\right).$$

98. Find the square root of

$$a^4+b^4-a^3b-ab^3+\frac{9a^2b^2}{4}.$$

99. Solve the equations:

(1) $\dfrac{3x}{x-1}-\dfrac{2x}{2x-1}=2$; (2) $3x^2+22x=493$.

100. Simplify $\dfrac{x(x+3y)-3(x+y)+2y^2}{2(x+y-1)-(1+x)}$.

101. What value of x will make the sum of $\dfrac{x-a}{2(a-b)}$ and $\dfrac{x+5b}{3(a+b)}$ equal to 2?

102. A man drives to a certain place at the rate of 8 miles an hour; returning by a road 3 miles longer at the rate of 9 miles an hour he takes $7\frac{1}{2}$ minutes longer than in going: how long is each road?

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103. Find the product of

(1) $3x^2-4xy+7y^2$, $3x^2+4xy+7y^2$;

(2) x^2-2y^2, $x^2-2xy+2y^2$, x^2+2y^2, $x^2+2xy+2y^2$.

104. Extract the square root of

$$1-\frac{3}{2}x^2+2x^3+\frac{9}{16}x^4-\frac{3}{2}x^5+x^6.$$

105. Find the highest factor common to

$$x(6x^2 - 8y^2) - y(3x^2 - 4y^2) \text{ and } 2xy(2y - x) + 4x^3 - 2y^3.$$

106. The sum of the digits of a number is 9, and if five times the digit in the tens' place be added to twice the digit in the units' place, the number will be inverted. What is the number?

107. If $\left(a + \frac{1}{a}\right)^2 = 3$, prove that $a^3 + \frac{1}{a^3} = 0$.

108. Solve the equations:

(1) $(x+7)(y-3)+2-(y+3)(x-1) = 5x - 11y + 35 = 0$;

(2) $\frac{1}{3} - \frac{7x-1}{6\frac{1}{2} - 3x} = \frac{8x-4}{3(x-2)}$.

---

109. If $x = b+c$, $y = c-a$, $z = a-b$, find the value of

$$x^2 + y^2 + z^2 - 2xy - 2xz + 2yz.$$

110. Express in the simplest form

$$\frac{1}{2x^2+3x+1} + \frac{1}{6x^2+5x+1} + \frac{1}{12x^2+7x+1} + \frac{1}{20x^2+9x+1}.$$

111. Resolve into factors:

(1) $4a^2b^2 - (a^2+b^2-c^2)^2$;

(2) $ab(m^2+1) + m(a^2+b^2)$.

112. Simplify the fractions:

(1) $\dfrac{1}{x + \dfrac{1}{1 + \dfrac{x+1}{3-x}}}$; (2) $\dfrac{a-x}{a^2 - ax - \dfrac{(a-x)^2}{1 - \dfrac{a}{x}}}$.

113. Solve the equations:

(1) $\frac{75-x}{3(x+1)} + \frac{80x+21}{5(3x+2)} = \frac{23}{x+1} + 5$;

(2) $\sqrt{x+12} - \sqrt{x} = 6$.

114. Ten minutes after the departure of an express train a slow train is started, travelling on an average 20 miles less per hour, which reaches a station 250 miles distant $3\frac{1}{2}$ hours after the arrival of the express. Find the rate at which each train travels.

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115. Simplify

$$\frac{a^3-64a}{a^2-4} \div \left[\frac{2a^2+5a+2}{2a^2+9a+4} \div \left\{\frac{a^3-4a}{a^2+4a} \div \frac{a^2+7a-8}{a^2+a-2}\right\}\right].$$

116. Resolve into four factors

$$4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2.$$

117. When $a=4,\ b=-2,\ c=\frac{3}{2},\ d=-1$, find the numerical value

$$\sqrt[3]{4c^2-a(a-2b-d)}-\sqrt[3]{b^4c+11b^3d^2}.$$

118. Find the value of

$$(1)\quad a-\cfrac{1}{b+\cfrac{1}{a+\cfrac{ab}{a-b}}};\qquad (2)\quad \cfrac{2}{2x-3+\cfrac{3}{1-\cfrac{x}{x-6}}}.$$

119. Solve the equations :

(1) $150x^2=299x+2$; (2) $ax+by=ay-bx=a^2+b^2$.

120. A has 19 miles to walk. At the end of a quarter of an hour he is overtaken by B who walks half a mile per hour faster; by walking at the same rate as B for the remainder of the journey he arrives half an hour sooner than he expected. Find how long the journey occupied each man.

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